Appendices for "Counterfactual Analysis and Inference with Nonstationary Data"

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A Auxiliary Results

This appendix collects the building blocks to our main result (Theorem 1). The first one is well-known result since Phillips (1991) and it will be stated here without proof for completeness.

Proposition 1. *Let $\boldsymbol{S}_t = \sum_{j=1}^{t} \boldsymbol{z}_j$ be the partial sum of the sequence $\{\boldsymbol{z}_t\}_{t=1}^{\infty}$ of $(n \times 1)$ random vectors. Then, under Assumption 3, (a) $\boldsymbol{\Sigma} = \lim_{T\to\infty} T^{-1}\mathbb{E}(\boldsymbol{S}_T\boldsymbol{S}_T')$ exists and is positive definite and (b) $\boldsymbol{Z}_T(r) \equiv T^{-1/2}\boldsymbol{S}_{[rT]} \Rightarrow \boldsymbol{\Sigma}^{1/2}\boldsymbol{W}(r)$, where $[\cdot]$ denotes the integer part and $\boldsymbol{W}(\cdot)$ is a vector Wiener process on $[0,1]^n$.*

The implied convergence in Proposition 1(a) is a direct consequence of the stationarity assumption together with the mixing condition as shown by Ibragimov and Linnik (1971). Finally, Proposition 1(b) is a multivariate generalization of the univariate invariance principle (Durlauf and Phillips 1985).

The next proposition state the relation between estimator applied to the pre-intervention period and the same estimator applied to the transformed variables. It also define the pseudo-true parameter $\boldsymbol{\beta}_0$ in term of the DGP parameters. Specifically, the OLS estimator $\widehat{\boldsymbol{\beta}}$ of y_{1t} on $\boldsymbol{y}_{0t}$ and a constant is related to the OLS estimator $\widehat{\boldsymbol{\gamma}}$ of $\boldsymbol{z}_{1t}$ on $\boldsymbol{z}_{0t}$ and a constant by:

Proposition 2. *Let $\boldsymbol{H} := \boldsymbol{H}(r,\boldsymbol{\mu})$ to be the transformation defined in (3.4) for $0 \le r \le n$ and $\boldsymbol{\mu} \in \mathbb{R}^n$. Then, $\underset{2:n+1:2:n+1}{[\boldsymbol{D}]}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) = \widehat{\boldsymbol{\gamma}}$, where $\boldsymbol{D} := \boldsymbol{H}^{-1}$ and $\boldsymbol{\beta}_0 := \left(\underset{2:n+1:2:n+1}{[\boldsymbol{D}]}\right)^{-1} \underset{2:n+1\times 1}{[\boldsymbol{D}]}$ is given by*

$$\boldsymbol{\beta}_0 := \boldsymbol{\beta}_0(r,\boldsymbol{\mu}) := \begin{cases} \boldsymbol{0} & \text{if } r = 0 \text{ and } \boldsymbol{\mu} = \boldsymbol{0}, \\ \left(\boldsymbol{0}_{n-2}, \frac{\mu_1}{\mu_n}, 0\right)' & \text{if } r = 0 \text{ and } \boldsymbol{\mu} \neq \boldsymbol{0}, \\ (-\boldsymbol{\Gamma}', \alpha)' & \text{if } r = 1, \\ [\boldsymbol{\pi}', (1, -\boldsymbol{\pi}')(-\boldsymbol{\Gamma}', \alpha)]' & \text{if } 2 \le r \le n-1 \text{ and} \\ [\boldsymbol{\pi}', (1, -\boldsymbol{\pi}')\boldsymbol{h}(0), \delta]' & \text{if } r = n. \end{cases}$$

We can now state the asymptotic distribution of $\widehat{\boldsymbol{\gamma}}$, i.e., the OLS estimator in the transformed variables. For that, we first establish some more notation. For any zero mean vector process covariance-stationary $\{\boldsymbol{v}_t\}$, we define the following $(n \times n)$ non-random matrices:

$$
\begin{aligned}
\boldsymbol{\Omega}_0(\boldsymbol{v}) &:= \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}(\boldsymbol{v}_t\boldsymbol{v}_t'), \\
\boldsymbol{\Omega}_j(\boldsymbol{v}) &:= \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{t-j} \mathbb{E}(\boldsymbol{v}_s\boldsymbol{v}_t'), \quad j = 1, 2, \dots \\
\boldsymbol{\Omega}(\boldsymbol{v}) &:= \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left(\sum_{t=1}^{T}\boldsymbol{v}_t\sum_{t=1}^{T}\boldsymbol{v}_t'\right), \\
\boldsymbol{\Upsilon}(\boldsymbol{v}) &:= \lim_{T\to\infty} \frac{1}{T}\mathbb{V}\left(\sum_{t=1}^{T}[\boldsymbol{v}_t]_1\boldsymbol{v}_t\right),
\end{aligned}
\tag{A.1}
$$

if the limits and expectations exist.

Some particular cases of the result below can be found elsewhere in the time-series literature. See, for instance, Durlauf and Phillips (1985) and Phillips (1986a,b).

Proposition 3. *Suppose that Assumption 1 holds and that $\{\boldsymbol{v}_t := \boldsymbol{H}'(\boldsymbol{\varepsilon}_t', 0)'\}_{t=1}^{\infty}$ fulfills Assumption 3, where $\boldsymbol{H} := \boldsymbol{H}(r, \boldsymbol{\mu})$ is the transformation defined in (3.4). Then, as $T \longrightarrow \infty$,*

$$\boldsymbol{\Lambda}_{T_0}\widehat{\boldsymbol{\gamma}} \Rightarrow \boldsymbol{q}^*,$$

where $\boldsymbol{\Lambda}_{T_0} := \boldsymbol{\Lambda}_{T_0}(r, \boldsymbol{\mu}) := \mathsf{diag}[\boldsymbol{\lambda}_{T_0}(r, \boldsymbol{\mu})]$ with

$$
\boldsymbol{\lambda}_{T_0}(r, \boldsymbol{\mu}) := \begin{cases}
(\mathbb{1}n-1, T_0^{-1/2})' & \text{if } r = 0 \text{ and } \boldsymbol{\mu} = \mathbf{0} \\
(\mathbb{1}n-2, T_0^{1/2}, T_0^{-1/2})' & \text{if } r = 0 \text{ and } \boldsymbol{\mu} \neq \mathbf{0} \\
(T_0\mathbb{1}n-1, T_0^{1/2})' & \text{if } r = 1 \text{ and } \boldsymbol{\mu} = \mathbf{0} \\
(T_0\mathbb{1}n-2, T_0^{3/2}, T_0^{1/2})' & \text{if } r = 1 \text{ and } \boldsymbol{\mu} \neq \mathbf{0} \\
(T_0^{1/2}\mathbb{1}r-1, T_0\mathbb{1}n-r, T_0^{1/2})' & \text{if } 2 \leq r \leq n-2 \text{ and } \boldsymbol{\mu} = \mathbf{0} \\
(T_0^{1/2}\mathbb{1}r-1, T_0\mathbb{1}n-1-r, T_0^{3/2}, T_0^{1/2})' & \text{if } 2 \leq r \leq n-2 \text{ and } \boldsymbol{\mu} \neq \mathbf{0} \\
(T_0^{1/2}\mathbb{1}n-1, T_0^{1/2})' & \text{if } r = n-1 \text{ and } \boldsymbol{\mu} = \mathbf{0} \\
(T_0^{1/2}\mathbb{1}n-2, T_0^{3/2}, T_0^{1/2})' & \text{if } r = n-1 \text{ and } \boldsymbol{\mu} \neq \mathbf{0} \\
(T_0^{1/2}\mathbb{1}n-2, F_{T_0}, T_0^{1/2})' & \text{if } r = n.
\end{cases}
$$

F_{T_0} is a positive increasing sequence such that $\lim_{T_0\to\infty}\sum_{t=1}^{T_0}\left(\frac{f_t}{F_{T_0}}\right)^2 =: \phi^2 > 0$ is finite and the

random vector $\boldsymbol{q}^* := \boldsymbol{q}^*(r,\boldsymbol{\mu})$ *is given by:*

$$\boldsymbol{q}^*(0,\boldsymbol{\mu}) := \begin{cases} \begin{pmatrix} \underset{2:n\times 2:n}{\left[\int_0^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{2:n}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}s\right]} \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} \underset{2:n\times 1:1}{\left[\int_0^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} \\ \underset{1:1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}s\right]} \end{pmatrix} & \text{, } \textit{if } \boldsymbol{\mu}=\boldsymbol{0} \\ \begin{pmatrix} \underset{2:n-1\times 2:n-1}{\left[\int_0^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{2:n-1}{\left[\mu_n\int_0^1 s\boldsymbol{B}\mathrm{d}s\right]} & \underset{2:n-1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}s\right]} \\ & \frac{\mu_n^2}{3} & \frac{\mu_n}{2} \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} \underset{2:n-1\times 1:1}{\left[\int_0^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} \\ \underset{1:1}{\left[\mu_n\int_0^1 s\boldsymbol{B}\mathrm{d}s\right]} \\ \underset{1:1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}s\right]} \end{pmatrix} & \text{, } \textit{if } \boldsymbol{\mu}\neq\boldsymbol{0} \end{cases}$$

$$\boldsymbol{q}^*(1,\boldsymbol{\mu}) := \begin{cases} \begin{pmatrix} \underset{2:n\times 2:n}{\left[\int_0^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{2:n}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}s\right]} \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} \underset{2:n\times 1:1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}\boldsymbol{B}'+\boldsymbol{\Omega}_0+\boldsymbol{\Omega}_1\right]} \\ \underset{1:1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}\right]} \end{pmatrix} & \text{, } \textit{if } \boldsymbol{\mu}=\boldsymbol{0} \\ \begin{pmatrix} \underset{2:n-1\times 2:n-1}{\left[\int_0^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{2:n-1}{\left[\mu_n\int_0^1 s\boldsymbol{B}\mathrm{d}s\right]} & \underset{2:n-1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}s\right]} \\ & \frac{\mu_n^2}{3} & \frac{\mu_n}{2} \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} \underset{2:n-1\times 1:1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}\boldsymbol{B}'+\boldsymbol{\Omega}_0+\boldsymbol{\Omega}_1\right]} \\ \underset{1:1}{\left[\mu_n\int_0^1 s\mathrm{d}\boldsymbol{B}\right]} \\ \underset{1:1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}\right]} \end{pmatrix} & \text{, } \textit{if } \boldsymbol{\mu}\neq\boldsymbol{0} \end{cases}$$

$$\boldsymbol{q}^*(r,\boldsymbol{\mu}) := \begin{cases} \begin{pmatrix} \underset{2:r\times 2:r}{[\boldsymbol{\Omega}_0]} & 0 & 0 \\ & \underset{r+1:n\times r+1:n}{\left[\int_0^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{r+1:n}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}s\right]} \\ & & 1 \end{pmatrix}^{-1} \times \begin{pmatrix} \underset{2:r}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}^*\right]} \\ \underset{r+1:n\times 1:1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}\boldsymbol{B}'+1(\boldsymbol{\Omega}_0+\boldsymbol{\Omega}_1)\right]} \\ \underset{1:1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}\right]} \end{pmatrix} & \text{, } \textit{if } 2\leq r\leq n-2 \textit{ and } \boldsymbol{\mu}=\boldsymbol{0}, \\ \begin{pmatrix} \underset{2:r\times 2:r}{[\boldsymbol{\Omega}_0]} & 0 & 0 & 0 \\ & \underset{r+1:n-1\times r+1:n-1}{\left[\int_0^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{r+1:n-1}{\left[\mu_n\int_0^1 s\boldsymbol{B}\mathrm{d}s\right]} & \underset{r+1:n-1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}s\right]} \\ & & \frac{\mu_n^2}{3} & \frac{\mu_n}{2} \\ & & & 1 \end{pmatrix}^{-1} \times \begin{pmatrix} \underset{2:r}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}^*\right]} \\ \underset{r+1:n-1\times 1:1}{\left[\int_0^1 \boldsymbol{B}\mathrm{d}\boldsymbol{B}'+1(\boldsymbol{\Omega}_0+\boldsymbol{\Omega}_1)\right]} \\ \underset{1:1}{\left[\mu_n\int_0^1 s\mathrm{d}\boldsymbol{B}\right]} \\ \underset{1:1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}\right]} \end{pmatrix} & \text{, } \textit{if } 2\leq r\leq n-2 \textit{ and } \boldsymbol{\mu}\neq\boldsymbol{0}, \end{cases}$$

$$\boldsymbol{q}^*(n-1,\boldsymbol{\mu}) := \begin{cases} \begin{pmatrix} \underset{2:n\times 2:n}{[\boldsymbol{\Omega}_0]} & 0 \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} \underset{2:n}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}^*\right]} \\ \underset{1:1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}\right]} \end{pmatrix} & \textit{, if } \boldsymbol{\mu}=\boldsymbol{0} \\ \begin{pmatrix} \underset{2:n-1\times 2:n-1}{[\boldsymbol{\Omega}_0]} & 0 & 0 \\ & \frac{\mu_n^2}{3} & \frac{\mu_n}{2} \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} \underset{2:n-1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}^*\right]} \\ \underset{1:1}{\left[\mu_n \int_0^1 s\mathrm{d}\boldsymbol{B}\right]} \\ \underset{1:1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}\right]} \end{pmatrix} & \textit{, if } \boldsymbol{\mu}\neq\boldsymbol{0} \end{cases}$$

$$\boldsymbol{q}^*(n,\boldsymbol{\mu}) := \begin{cases} \begin{pmatrix} \underset{2:n\times 2:n}{[\boldsymbol{\Omega}_0]} & 0 \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} \underset{2:n}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}^*\right]} \\ \underset{1:1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}\right]} \end{pmatrix} & \textit{, for } \boldsymbol{\mu}=\boldsymbol{0} \\ \begin{pmatrix} \underset{2:n-1\times 2:n-1}{[\boldsymbol{\Omega}_0]} & 0 & 0 \\ & \mu_n^2\phi^2 & \mu_n\widetilde{\phi} \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} \underset{2:n-1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}^*\right]} \\ \underset{1:1}{[\mu_n \phi \boldsymbol{B}]} \\ \underset{1:1}{\left[\int_0^1 \mathrm{d}\boldsymbol{B}\right]} \end{pmatrix} & \textit{, for } \boldsymbol{\mu}\neq\boldsymbol{0}, \end{cases}$$

with $\widetilde{\phi} := \lim_{T_0\to\infty} \sum_{t=1}^{T_0} \frac{f_t}{F_{T_0}\sqrt{T_0}} < \infty$, $\boldsymbol{B} := \boldsymbol{\Omega}^{1/2}\boldsymbol{W}$ *and* $\boldsymbol{B}^* := \boldsymbol{\Upsilon}^{1/2}\boldsymbol{W}$ *where* $\boldsymbol{W} := \{\boldsymbol{W}(s), s \in [0,1]\}$, *denotes a standard vector Wiener process on* $[0,1]^n$ *and* $\boldsymbol{\Omega} := \boldsymbol{\Omega}(\boldsymbol{v})$, $\boldsymbol{\Omega}_0 := \boldsymbol{\Omega}_0(\boldsymbol{v})$, $\boldsymbol{\Omega}_1 := \boldsymbol{\Omega}_1(\boldsymbol{v})$, $\boldsymbol{\Upsilon} := \boldsymbol{\Upsilon}(v)$ *are defined in* (A.1).

If the DGP has only deterministic trends ($r = n$), the trend is of the form $f_t = t^k, k > 0$. In this case, $F_{T_0} = T_0^{k+1/2}$ and $\phi^2 := \lim_{T_0\to\infty} \frac{1}{T_0}\sum_{t=1}^{T_0}\left(\frac{t}{T_0}\right)^{2k} = \frac{1}{2k+1}$ and $\widetilde{\phi} := \lim_{T\to\infty} \sum_{t=1}^{T_0} \frac{t^k}{T^{k+1}} = \frac{1}{k+1}$. For $k = 1$, the asymptotic distributions for the DGP when $r = n-1$ and $r = n$ are identical.

B Unpacking Notation of Theorem 1

Set $V := V(r) := \underset{1:1\times 1:1}{[\boldsymbol{\Omega}_0]}\, I(r > 0)$, where $I(\cdot)$ is the indicator function. Define $\boldsymbol{G} := \boldsymbol{G}(r) := \lim_{T_0\to\infty} \xi_{T_0}\left(\boldsymbol{\Lambda}_T \underset{2:n+1:2:n+1}{[\boldsymbol{D}]}\right)^{-1}$. The $n+1$-dimensional random vector $\boldsymbol{p} := \boldsymbol{p}(r,\boldsymbol{\mu})$, for the case where $\boldsymbol{\mu} = \boldsymbol{0}$ is given by

$$\boldsymbol{p}(0,\boldsymbol{0}) := \begin{bmatrix} \int_{\lambda_0}^1 \boldsymbol{B}\mathrm{d}s \\ 1-\lambda_0 \end{bmatrix}; \quad \boldsymbol{p}(1,\boldsymbol{0}) := \begin{bmatrix} \underset{1:1}{\left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]} \\ \underset{2:n}{\left[\int_{\lambda_0}^1 \boldsymbol{B}\mathrm{d}s\right]} \\ 1-\lambda_0 \end{bmatrix}; \quad \boldsymbol{p}(r,\boldsymbol{0}) := \begin{bmatrix} \underset{1:1}{\left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]} \\ \underset{2:r}{[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}]} \\ \underset{r+1:n}{[\int_{\lambda_0}^1 \boldsymbol{B}\mathrm{d}s]} \\ 1-\lambda_0 \end{bmatrix} \quad \text{for } 2 \le r \le n-2;$$

$$\boldsymbol{p}(n-1,\boldsymbol{0}) = \boldsymbol{p}(n,\boldsymbol{0}) := \begin{bmatrix} \underset{1:1}{\left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]} \\ \underset{2:n}{\left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]} \\ 1-\lambda_0 \end{bmatrix},$$

and for $\boldsymbol{\mu} \neq 0$:

$$\boldsymbol{p}(0,\boldsymbol{\mu}) := \begin{bmatrix} \left[\int_{\lambda_0}^1 \boldsymbol{B}\mathrm{d}s\right]_{1:n-1} \\ \frac{\mu_n(1-\lambda_0^2)}{2} \\ 1-\lambda_0 \end{bmatrix}; \quad \boldsymbol{p}(1,\boldsymbol{\mu}) := \begin{bmatrix} \left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]_{1:1} \\ \left[\int_{\lambda_0}^1 \boldsymbol{B}\mathrm{d}s\right]_{2:n-1} \\ \frac{\mu_n(1-\lambda_0^2)}{2} \\ 1-\lambda_0 \end{bmatrix}; \quad \boldsymbol{p}(r,\boldsymbol{\mu}) := \begin{bmatrix} \left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]_{1:1} \\ \left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]_{2:r} \\ \left[\int_{\lambda_0}^1 \boldsymbol{B}\mathrm{d}s\right]_{r+1:n-1} \\ \frac{\mu_n(1-\lambda_0^2)}{2} \\ 1-\lambda_0 \end{bmatrix} \quad \text{for } 2 \leq r \leq n-2;$$

$$\boldsymbol{p}(n-1,\boldsymbol{\mu}) := \begin{bmatrix} \left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]_{1:1} \\ \left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]_{2:n-1} \\ \frac{\mu_n(1-\lambda_0^2)}{2} \\ 1-\lambda_0 \end{bmatrix}; \qquad \boldsymbol{p}(n,\boldsymbol{\mu}) := \begin{bmatrix} \left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]_{1:1} \\ \left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}\right]_{2:n-1} \\ \mu_n\check{\phi} \\ 1-\lambda_0 \end{bmatrix},$$

where $\check{\phi} := \lim_{T\to\infty} \sum_{t>T_0}^{T} \frac{f_t}{F_T\sqrt{T}} < \infty$ and the processes $\boldsymbol{B}$ will be defined below.

The $n+1$-dimensional random vector $\boldsymbol{q} := \boldsymbol{q}(r,\boldsymbol{\mu})$ is defined as

$$\boldsymbol{q}(0,\boldsymbol{\mu}) := \begin{cases} \begin{pmatrix} \left[\int_0^{\lambda_0} \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]_{2:n\times 2:n} & \left[\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}s\right]_{2:n} \\ & \lambda_0 \end{pmatrix}^{-1} \begin{pmatrix} \left[\int_0^{\lambda_0} \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]_{2:n\times 1:1} \\ \left[\int_0^{\lambda} \boldsymbol{B}\mathrm{d}s\right]_{1:1} \end{pmatrix} & \text{, if } \boldsymbol{\mu} = \boldsymbol{0} \\ \begin{pmatrix} [\int_0^{\lambda_0} BB'\mathrm{d}s]_{2:n-1\times 2:n-1} & [\mu_n \int_0^{\lambda_0} sB\mathrm{d}s]_{2:n-1} & [\int_0^{\lambda_0} B\mathrm{d}s]_{2:n-1} \\ & \frac{\mu_n^2\lambda_0^3}{3} & \frac{\mu_n\lambda_0^2}{2} \\ & & \lambda_0 \end{pmatrix}^{-1} \begin{pmatrix} [\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}\boldsymbol{B}' + \lambda_0(\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_1)]_{2:n-1\times 1:1} \\ \left[\mu_n \int_0^{\lambda_0} s\mathrm{d}\boldsymbol{B}\right]_{1:1} \\ \left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}\right]_{1:1} \end{pmatrix} & \text{, if } \boldsymbol{\mu} \neq \boldsymbol{0} \end{cases}$$

$$\boldsymbol{q}(1,\boldsymbol{\mu}) := \begin{cases} \begin{pmatrix} [\int_0^{\lambda_0} \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s]_{2:n\times 2:n} & [\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}s]_{2:n} \\ & \lambda_0 \end{pmatrix}^{-1} \begin{pmatrix} [\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}\boldsymbol{B}' + \boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_1]_{2:n\times 1:1} \\ \left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}\right]_{1:1} \end{pmatrix} & \text{, if } \boldsymbol{\mu} = \boldsymbol{0} \\ \begin{pmatrix} [\int_0^{\lambda_0} BB'\mathrm{d}s]_{2:n-1\times 2:n-1} & [\mu_n \int_0^{\lambda_0} sB\mathrm{d}s]_{2:n-1} & [\int_0^{\lambda_0} B\mathrm{d}s]_{2:n-1} \\ & \frac{\mu_n^2\lambda_0^3}{3} & \frac{\mu_n\lambda_0^2}{2} \\ & & \lambda_0 \end{pmatrix}^{-1} \begin{pmatrix} [\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}\boldsymbol{B}' + \lambda_0(\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_1)]_{2:n-1\times 1:1} \\ \left[\mu_n \int_0^{\lambda_0} s\mathrm{d}\boldsymbol{B}\right]_{1:1} \\ \left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}\right]_{1:1} \end{pmatrix} & \text{, if } \boldsymbol{\mu} \neq \boldsymbol{0} \end{cases}$$

$$
\boldsymbol{q}(r,\boldsymbol{\mu}) := \begin{cases}
\begin{pmatrix}
\underset{2:r\times 2:r}{[\boldsymbol{\Omega}_0]} & 0 & 0 \\
 & \underset{r+1:n\times r+1:n}{\left[\int_0^{\lambda_0} \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{r+1:n}{\left[\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}s\right]} \\
 & & \lambda_0
\end{pmatrix}^{-1}
\times \begin{pmatrix}
\underset{2:r}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}^*\right]} \\
\underset{r+1:n\times 1:1}{\left[\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}\boldsymbol{B}' + 1(\boldsymbol{\Omega}_0+\boldsymbol{\Omega}_1)\right]} \\
\underset{1:1}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}\right]}
\end{pmatrix} & \text{, if } 2\le r\le n-2 \text{ and } \boldsymbol{\mu}=\boldsymbol{0}, \\
\begin{pmatrix}
\underset{2:r\times 2:r}{[\Omega_0]} & 0 & 0 & 0 \\
 & \underset{r+1:n-1\times r+1:n-1}{\left[\int_0^{\lambda_0} \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{r+1:n-1}{\left[\mu_n\int_0^{\lambda_0} s\boldsymbol{B}\mathrm{d}s\right]} & \underset{r+1:n-1}{\left[\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}s\right]} \\
 & & \frac{\mu_n^2\lambda_0^3}{3} & \frac{\mu_n\lambda_0^2}{2} \\
 & & & \lambda_0
\end{pmatrix}^{-1}
\times \begin{pmatrix}
\underset{2:r}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}^*\right]} \\
\underset{r+1:n-1\times 1:1}{\left[\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}\boldsymbol{B}' + \lambda_0(\boldsymbol{\Omega}_0+\boldsymbol{\Omega}_1)\right]} \\
\underset{1:1}{\left[\mu_n\int_0^{\lambda_0} s\mathrm{d}\boldsymbol{B}\right]} \\
\underset{1:1}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}\right]}
\end{pmatrix} & \text{, if } 2\le r\le n-2 \text{ and } \boldsymbol{\mu}\neq\boldsymbol{0},
\end{cases}
$$

$$
\boldsymbol{q}(n-1,\boldsymbol{\mu}) := \begin{cases}
\begin{pmatrix}
\underset{2:n\times 2:n}{[\boldsymbol{\Omega}_0]} & 0 \\
 & \lambda_0
\end{pmatrix}^{-1}
\begin{pmatrix}
\underset{2:n}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}^*\right]} \\
\underset{1:1}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}\right]}
\end{pmatrix} & \text{, if } \boldsymbol{\mu}=\boldsymbol{0} \\
\begin{pmatrix}
\underset{2:n-1\times 2:n-1}{[\boldsymbol{\Omega}_0]} & 0 & 0 \\
 & \frac{\mu_n^2\lambda_0^3}{3} & \frac{\mu_n\lambda_0^2}{2} \\
 & & \lambda_0
\end{pmatrix}^{-1}
\begin{pmatrix}
\underset{2:n-1}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}^*\right]} \\
\underset{1:1}{\left[\mu_n\int_0^{\lambda_0} s\mathrm{d}\boldsymbol{B}\right]} \\
\underset{1:1}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}\right]}
\end{pmatrix} & \text{, if } \boldsymbol{\mu}\neq\boldsymbol{0}
\end{cases}
$$

$$
\boldsymbol{q}(n,\boldsymbol{\mu}) := \begin{cases}
\begin{pmatrix}
\underset{2:n\times 2:n}{[\boldsymbol{\Omega}_0]} & 0 \\
 & \lambda_0
\end{pmatrix}^{-1}
\begin{pmatrix}
\underset{2:n}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}^*\right]} \\
\underset{1:1}{\left[\int_0^{\lambda} \mathrm{d}\boldsymbol{B}\right]}
\end{pmatrix} & \text{, for } \boldsymbol{\mu}=\boldsymbol{0} \\
\begin{pmatrix}
\underset{2:n-1\times 2:n-1}{[\boldsymbol{\Omega}_0]} & 0 & 0 \\
 & \mu_n^2\phi^2 & \mu_n\widetilde{\phi} \\
 & & \lambda_0
\end{pmatrix}^{-1}
\begin{pmatrix}
\underset{2:n-1}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}^*\right]} \\
\underset{1:1}{[\mu_n\phi\boldsymbol{B}]} \\
\underset{1:1}{\left[\int_0^{\lambda_0} \mathrm{d}\boldsymbol{B}\right]}
\end{pmatrix} & \text{, for } \boldsymbol{\mu}\neq\boldsymbol{0},
\end{cases}
$$

Finally, the random variable $a := a(r, \boldsymbol{\mu})$, for the case when $\boldsymbol{\mu} = \boldsymbol{0}$, is given by

$$a(r,\boldsymbol{0}) := \begin{cases} \frac{1}{1-\lambda_0}\left(\underset{1:1\times 1:1}{\left[\int_{\lambda_0}^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} - 2\boldsymbol{q}(0,\boldsymbol{0})' \begin{bmatrix} \underset{2:n\times 1:1}{\left[\int_{\lambda_0}^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} \\ \underset{1:1}{\left[\int_{\lambda_0}^1 \boldsymbol{B}\mathrm{d}s\right]} \end{bmatrix} \right. \\ \qquad \left. + \boldsymbol{q}(0,\boldsymbol{0})' \begin{bmatrix} \underset{2:n\times 2:n}{\left[\int_0^{\lambda_0} \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{2:n}{\left[\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}s\right]} \\ & \lambda_0 \end{bmatrix} \boldsymbol{q}(0,\boldsymbol{0}) \right) & \text{, if } r = 0 \\ \frac{1}{1-\lambda_0} \underset{1:1}{\left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}^*\right]} & \text{, for } r \geq 1 \end{cases}$$

and for the case $\boldsymbol{\mu} \neq 0$:

$$a(r,\boldsymbol{\mu}) := \begin{cases} \frac{1}{1-\lambda_0}\left(\underset{1:1\times 1:1}{\left[\int_{\lambda_0}^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} - 2\boldsymbol{q}(0,\boldsymbol{\mu})' \begin{bmatrix} \underset{2:n-1\times 1:1}{\left[\int_{\lambda_0}^1 \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} \\ \mu_n \underset{1:1}{\left[\int_{\lambda_0}^1 s\boldsymbol{B}\mathrm{d}s\right]} \\ \underset{1:1}{\left[\int_{\lambda_0}^1 \boldsymbol{B}\mathrm{d}s\right]} \end{bmatrix} \right. \\ \qquad \left. + \boldsymbol{q}(0,\boldsymbol{\mu})' \begin{bmatrix} \underset{2:n-1\times 2:n-1}{\left[\int_0^{\lambda_0} \boldsymbol{B}\boldsymbol{B}'\mathrm{d}s\right]} & \underset{2:n-1}{\left[\mu_n \int_0^{\lambda_0} s\boldsymbol{B}\mathrm{d}s\right]} & \underset{2:n-1}{\left[\int_0^{\lambda_0} \boldsymbol{B}\mathrm{d}s\right]} \\ & \frac{\mu_n^2 \lambda_0^3}{3} & \frac{\mu_n \lambda_0^2}{2} \\ & & \lambda_0 \end{bmatrix} \boldsymbol{q}(0,\boldsymbol{\mu}) \right) & \text{, if } r = 0 \\ \frac{1}{1-\lambda_0} \underset{1:1}{\left[\int_{\lambda_0}^1 \mathrm{d}\boldsymbol{B}^*\right]} & \text{, for } r \geq 1, \end{cases}$$

with $\widetilde{\phi} := \lim_{T\to\infty} \sum_{t=1}^{T_0} \frac{f_t}{F_T\sqrt{T}} < \infty$, $\boldsymbol{B} := \boldsymbol{\Omega}^{1/2}\boldsymbol{W}$ and $\boldsymbol{B}^* := \boldsymbol{\Upsilon}^{1/2}\boldsymbol{W}$ where $\boldsymbol{W} := \{\boldsymbol{W}(s), s \in [0,1]\}$, denotes a standard vector Wiener process on $[0,1]^n$ and $\boldsymbol{\Omega} := \boldsymbol{\Omega}(v)$, $\boldsymbol{\Omega}_0 := \boldsymbol{\Omega}_0(v)$, $\boldsymbol{\Omega}_1 := \boldsymbol{\Omega}_1(v)$, $\boldsymbol{\Upsilon} := \boldsymbol{\Upsilon}(v)$ are defined in (A.1).

C Proof of the Main Results

Proof of Proposition 2

Let $\widetilde{\boldsymbol{y}}_t := (\boldsymbol{y}_t^{(0)'}, 1)'$ and recall that $\boldsymbol{z}_t := \boldsymbol{H}'\widetilde{\boldsymbol{y}}_t$. Let $\widetilde{\boldsymbol{Y}}$ be the $(T_0 \times n+1)$ matrix constructed by stacking $\widetilde{\boldsymbol{y}}_t'$ for $t = 1, \ldots, T_0$. Define $\boldsymbol{Z}$ by stacking $\boldsymbol{z}_t'$. Hence, $\boldsymbol{Z} = \widetilde{\boldsymbol{Y}}\boldsymbol{H}$. Define the $(n+1 \times n+1)$ matrices $\boldsymbol{\Sigma} := \widetilde{\boldsymbol{Y}}'\widetilde{\boldsymbol{Y}}$ and $\boldsymbol{\Omega} := \boldsymbol{Z}'\boldsymbol{Z} =$, such that

$$\widehat{\boldsymbol{\beta}} := \left(\underset{2:n+1\times 2:n+1}{[\boldsymbol{\Sigma}]} \right)^{-1} \underset{2:n+1\times 1:1}{[\boldsymbol{\Sigma}]} \quad \text{and} \quad \widehat{\boldsymbol{\gamma}} := \left(\underset{2:n+1\times 2:n+1}{[\boldsymbol{\Omega}]} \right)^{-1} \underset{2:n+1\times 1:1}{[\boldsymbol{\Omega}]},$$

provided that $\underset{2:n+1\times 2:n+1}{[\boldsymbol{\Sigma}]}$ and $\underset{2:n+1\times 2:n+1}{[\boldsymbol{\Omega}]}$ are non-singular. To show the relation between $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\gamma}}$ recall that $\boldsymbol{D} = \boldsymbol{H}^{-1}$. Therefore, we may write $\boldsymbol{\Sigma} = \boldsymbol{D}'\boldsymbol{\Omega}\boldsymbol{D}$. Notice that

$$\underset{2:n+1\times 2:n+1}{[\boldsymbol{\Sigma}]} = \left(\underset{1:1\times 2:n+1}{[\boldsymbol{D}]}' \quad \underset{2:n+1\times 2:n+1}{[\boldsymbol{D}]}'\right)\boldsymbol{\Omega}\begin{pmatrix}\underset{1:1\times 2:n+1}{[\boldsymbol{D}]}\\ \underset{2:n+1\times 2:n+1}{[\boldsymbol{D}]}\end{pmatrix}$$

$$\underset{2:n+1\times 1:1}{[\boldsymbol{\Sigma}]} = \left(\underset{1:1\times 2:n+1}{[\boldsymbol{D}]}' \quad \underset{2:n+1\times 2:n+1}{[\boldsymbol{D}]}'\right)\boldsymbol{\Omega}\begin{pmatrix}\underset{1:1\times 1:1}{[\boldsymbol{D}]}\\ \underset{2:n+1\times 1:1}{[\boldsymbol{D}]}\end{pmatrix}$$

In all cases considered we have that $\underset{1:1\times 2:n+1}{[\boldsymbol{D}]} = \boldsymbol{0}$, which implies that we can rewrite $\widehat{\boldsymbol{\beta}}$ as

$$\widehat{\boldsymbol{\beta}} = \left(\underset{2:n+1\times 2:n+1}{[\boldsymbol{D}]}\right)^{-1}(\widehat{\boldsymbol{\gamma}} + \underset{2:n+1\times 1:1}{[\boldsymbol{D}]})$$

Rearranging the terms and setting $\boldsymbol{\beta}_0 = \left(\underset{2:n+1\times 2:n+1}{[\boldsymbol{D}]}\right)^{-1}\underset{2:n+1\times 1:1}{[\boldsymbol{D}]}$ yield the result.

Proof of Proposition 3

Let $z_{1t} := \underset{1:1}{[\boldsymbol{z}_t]}$ and $\boldsymbol{z}_{0t} := \underset{2:n+1}{[\boldsymbol{z}_t]}$. Then,

$$\boldsymbol{\Lambda}_T\widehat{\boldsymbol{\gamma}} = \left(\boldsymbol{\Lambda}_T^{-1}\sum_{t=1}^{T_0}\boldsymbol{z}_{0t}\boldsymbol{z}_{0t}'\boldsymbol{\Lambda}_T^{-1}\right)^{-1}\boldsymbol{\Lambda}_T^{-1}\sum_{t=1}^{T_0}\boldsymbol{z}_{0t}z_{1t} =: \boldsymbol{M}_T^{-1}\boldsymbol{m}_T.$$

Applying the convergence results of Lemma 1 and the continuous mapping theorem, we have $\boldsymbol{M}_T^{-1}\boldsymbol{m}_T \Rightarrow \boldsymbol{M}^{-1}\boldsymbol{m} =: \boldsymbol{q}^*$, where the non-singular random matrix $\boldsymbol{M} := \boldsymbol{M}(r,\boldsymbol{\mu})$ and the random vector $\boldsymbol{m} := \boldsymbol{m}(r,\boldsymbol{\mu})$ are defined in the Proposition 3.

Proof of Theorem 1

First notice that $\widehat{\delta}_t - \delta_t = \nu_t - (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)'\boldsymbol{x}_t = z_{1t} - \widehat{\boldsymbol{\gamma}}'\boldsymbol{z}_{0t}$. Therefore, for (a) we have

$$\begin{aligned}\xi_{T_0}(\widehat{\delta}_t - \delta_t - \nu_t) &= -\xi_{T_0}\boldsymbol{x}_t'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) = -\xi_{T_0}\boldsymbol{x}_t'\left(\underset{2:n+1:2:n+1}{[\boldsymbol{D}]}\right)^{-1}\widehat{\boldsymbol{\gamma}}\\ &= -\boldsymbol{x}_t'\left[\xi_{T_0}\left(\underset{2:n+1:2:n+1}{[\boldsymbol{D}]}\right)^{-1}\boldsymbol{\Lambda}_T^{-1}\right](\boldsymbol{\Lambda}_T\widehat{\boldsymbol{\gamma}}) \Rightarrow -(\boldsymbol{G}\boldsymbol{q})'\boldsymbol{x}_t,\end{aligned}$$

where the convergence in distribution follows from the definition of $\boldsymbol{G}$, Proposition 3 and the Continuous Mapping Theorem (CMT). For (b), Lemma 1 and the CMT yield

$$\xi_T(\widehat{\Delta}_T - \Delta_T) = \frac{T}{T_2}\left[\left(\frac{\xi_T}{T}\sum_{t>T_0}^{T} z_{1t}\right) - (\boldsymbol{\Lambda}_T\widehat{\boldsymbol{\gamma}})'\left(\frac{\xi_T}{T}\boldsymbol{\Lambda}_T^{-1}\sum_{t>T_0}^{T}\boldsymbol{z}_{0t}\right)\right] \Rightarrow \frac{1}{1-\lambda_0}(1,\boldsymbol{q}')\boldsymbol{p},$$

where the random vectors $\boldsymbol{q}$ and $\boldsymbol{p}$ are defined as in the Theorem 1 for each case of r and $\boldsymbol{\mu}$. Similarly, for (c) we have that $(\widehat{\delta}_t - \delta_t)^2 = z_{1t}^2 - 2\widehat{\boldsymbol{\gamma}}'\boldsymbol{z}_{0t}z_{1t} + \widehat{\boldsymbol{\gamma}}'\boldsymbol{z}_{0t}\boldsymbol{z}_{0t}'\widehat{\boldsymbol{\gamma}}$. Hence,

$$\zeta_T\widehat{V}_T = \frac{T}{T_2}\left[\left(\frac{\zeta_T}{T}\sum_{t>T_0}^{T} z_{1t}^2\right) - 2(\boldsymbol{\Lambda}_T\widehat{\boldsymbol{\gamma}})'\left(\frac{\zeta_T}{T}\boldsymbol{\Lambda}_T^{-1}\sum_{t>T_0}^{T}\boldsymbol{z}_{0t}z_{1t}\right)\right.$$
$$\left. + (\boldsymbol{\Lambda}_T\widehat{\boldsymbol{\gamma}})'\left(\frac{\zeta_T}{T}\boldsymbol{\Lambda}_T^{-1}\sum_{t>T_0}^{T}\boldsymbol{z}_{0t}\boldsymbol{z}_{0t}'\boldsymbol{\Lambda}_T^{-1}\right)(\boldsymbol{\Lambda}_T\widehat{\boldsymbol{\gamma}})\right].$$

For $r = 0$ and $\zeta_T = 1/T$ we have the following result

$$\frac{1}{T}\widehat{V}_T \Rightarrow \frac{1}{1-\lambda_0}\left[\boldsymbol{j}_1 - 2\boldsymbol{q}(0)'\boldsymbol{j}_2 + \boldsymbol{q}(0)'\boldsymbol{N}\boldsymbol{q}(0)\right] =: a,$$

where the random vectors $\boldsymbol{j}_1$, $\boldsymbol{j}_2$ and matrix $\boldsymbol{N}$ are defined in Appendix B. For $r \geq 1$, $\zeta_T = \sqrt{T}$ and the last two terms in parenthesis vanish in probability such that

$$\sqrt{T}\left(\widehat{V}_T - \underset{1:1\times 1:1}{[\boldsymbol{\Omega}_0]}\right) \Rightarrow \frac{1}{1-\lambda_0}\underset{1:1}{\left[\int_{\lambda_0}^{1} d\boldsymbol{B}^*\right]}.$$

Proof of Theorem 2

Part (a) follows directly from of Theorem 1(a) combined with the Continuous Mapping Theorem. For (b), let $\widetilde{G}_T(\boldsymbol{x}) := \frac{1}{\tau}\sum_{j=1}^{\tau} I(\boldsymbol{\psi}_j \leq \boldsymbol{x})$ be the unfeasible counterpart of $\widehat{G}_T$, where $\tau := T_0 - T_2 + 1$. We now show that both $\widetilde{G}_T(\boldsymbol{x}) - G_T(\boldsymbol{x})$ and $\widehat{G}_T(\boldsymbol{x}) - \widetilde{G}_T(\boldsymbol{x})$ vanish in probability as $T_0 \to \infty$. The result then follows by the triangle inequality.

Due to the strictly stationarity assumption $\mathbb{E}\widetilde{G}_T(x) = \frac{1}{\tau}\sum_{j=1}^{\tau}\mathbb{P}(\psi_j \leq x) = \mathbb{P}(\psi_0 \leq x) =: G_T(x)$. Hence, $\widetilde{G}_T(x)$ is unbiased for $G_T(x)$. So, it is enough to show that $\mathbb{V}\widetilde{G}_T(x)$ converges to zero. The sequence $\{W_j := I(\psi_j \leq x)\}_j$ is stationary and, as a consequence,

$$\mathbb{V}\widetilde{G}_T(x) = \frac{1}{\tau}\sum_{|k|<\tau}(1 - \frac{|k|}{\tau})\gamma_k, \quad \gamma_k := \mathbb{C}(W_1, W_{1+k}).$$

Also, $0 \leq W_j \leq 1$. We can bound the first $T_2 - 1$ covariances by 1 and the remanning ones using a mixing inequality due to **?**]. For $|k| \geq T_2$, we have $\gamma_k \leq 4\alpha(k - T_2 + 1)$, where $\alpha(j)$ is the mixing coefficient of $\{\epsilon_t\}_t$.[6] Then,

$$\mathbb{V}\widetilde{G}_T(x) \leq \frac{2T_2 + 1}{\tau} + \frac{8}{\tau}\sum_{k=T_2}^{\tau}\alpha(k - T_2 + 1).$$

Finally, since $T_0 \to \infty$ implies $\tau \to \infty$, we have the first term converging to zero and the second converges to zero due to the strong mixing assumption.

[6] In fact the sequence $\{W_j(\nu_j, \ldots, \nu_{j+T_2-1}\}_j$ is also strong mixing.

For the second part, fix $\boldsymbol{x}$ as a continuity point of G and write

$$\widehat{G}(\boldsymbol{x}) := \frac{1}{\tau}\sum_{j=1}^{\tau} I(\boldsymbol{\psi}_0 + (\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0) \leq \boldsymbol{x}).$$

For any $\epsilon > 0$ define the events $\mathscr{A}_T(\epsilon) = \{\|\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0\|_\infty < \epsilon\}$ and $\mathscr{B}_T(\epsilon, x) = \{|\widetilde{G}(x) - G(x)| < \epsilon\}$. On $\mathscr{A}_T$ we have $\widetilde{G}(\boldsymbol{x} - \epsilon\boldsymbol{\iota}) \leq \widehat{G}(\boldsymbol{x}) \leq \widetilde{G}(\boldsymbol{x} + \epsilon\boldsymbol{\iota})$, where $\boldsymbol{\iota}$ is a conformable vector of ones. If we condition on $\mathscr{B}_T$, we have for the continuity points of $G(x)$ $G(\boldsymbol{x} - \epsilon\boldsymbol{\iota}) - \epsilon \leq \widehat{G}(\boldsymbol{x}) \leq G(\boldsymbol{x} + \epsilon\boldsymbol{\iota}) + \epsilon$. Set $\epsilon \to 0$ to conclude since $\mathscr{A}_T \cap \mathscr{B}_T$ occurs with probability approaching 1.

For (c) we use the fact that (b) is equivalent (refer to Theorem 6.3.1 of **?**]) to say that for any subsequence $\{T_j\}$, we can extract a subsequence $\{T_{j_k}\}$ such that $\widehat{G}_{T_{j_k}}(\omega, x) \to G(x)$ for all $\omega \in \Omega_3$ and x a continuity point of G with $\mathbb{P}(\Omega_3) = 1$. Since G is assumed continuous and for each fixed ω, $\widehat{G}_{T_{j_k}}(\omega, x)$ is a CDF, the last convergence can be made uniform by Polya's theorem, i.e., $\sup_{x \in \mathbb{R}^b} |\widehat{G}_{T_{j_k}}(\omega, x) - G_{T_{j_k}}(x)| \to 0$ for all $\omega \in \Omega_3$. The result then follows by using the equivalence (in the other direction) of Theorem 6.3.1 of **?**].

Proof of Theorem 3

Let F denote the CDF of p, i.e. $F(x) := \mathbb{P}(p \leq x)$. Since $\sqrt{b}L_j$ and $\sqrt{T}(\widehat{\Delta}_T - \Delta_T)$ has the same limiting distribution p according to Theorem 1(b) and p is a continuous random variable, we have that $\widehat{F}_{t,b}$ is asymptotically mean unbiased for F_T for every $x \in \mathbb{R}$ since

$$\mathbb{E}(\widehat{F}_{T,b}(x) - F_T(x)) = \frac{1}{\#\mathcal{J}}\sum_{j\in\mathcal{J}} \mathbb{P}(\sqrt{b}L_j \leq x) - \mathbb{P}(\sqrt{T}(\widehat{\Delta}_T - \Delta_T) \leq x) = o(1).$$

To show that $\widehat{F}_{T,b}(x)$ converges to $\mathcal{F}_T(x)$ in probability, it is enough to show that the variance of $\widehat{F}_{T,b}(x)$ vanishes. Let $H_j := I(\sqrt{b}L_j \leq x)$ and $\gamma_{i,j} := \mathbb{C}(H_i, H_j)$. Since H_j is binary, we have that $|\gamma_{ij}| \leq 1$. Therefore, $\mathbb{V}(\widehat{F}_{T,b}(x)) \leq \frac{(2b+1)}{T-b+1} + \frac{1}{(T-b+1)^2}\sum_{|i-j|>b}\gamma_{i,j}$. The first term is $o(1)$ under the theorem's assumptions. For the second term, notice that for any pair $(i,j) \in \mathcal{J}^2$ we have that H_i and H_j are functions of the subsamples indexed by $\mathcal{S}_i$ and $\mathcal{S}_j$, respectively. For $|i-j| > b$ we have $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$. Hence, we can bound the covariance as $|\gamma_{i,j}| \leq 4\alpha(|i-j| - b)$, where $\alpha(\cdot)$ is the mixing coefficient of the process $\{\boldsymbol{\epsilon}_t\}_t$ and the mixing inequality is due to Ibragimov (1962). Thus,

$$\frac{1}{(T-b+1)^2}\sum_{|i-j|>b}\gamma_{i,j} \leq \frac{4}{(T-b+1)^2}\sum_{|i-j|>b}\alpha(|i-j|-b) \leq \frac{8}{(T-b+1)}\sum_{k=1}^{T-b+1}\alpha(k) = o(1),$$

which proves the pointwise convergence, namely $|F_{T,b}(x) - F_T(x)| \xrightarrow{p} 0$ for every $x \in \mathbb{R}$.

For the uniform result we once again use the equivalence given in Theorem 6.3.1 of Resnick (1999) to say that for any subsequence $\{T_j\}$, we can extract a subsequence $\{T_{j_k}\}$ such that $\widehat{F}_{T_{j_k}}(\omega, x) \to F(x)$ for all $\omega \in \Omega_4$ and $x \in \mathbb{R}$ with $\mathbb{P}(\Omega_4) = 1$. Since F is continuous and for each fixed ω, $\widehat{F}_{T_{j_k}}(\omega, x)$ is a CDF, the last convergence can be made uniform by Polya's theorem such that $\sup_{x \in \mathbb{R}^b} |\widehat{F}_{T_{j_k}}(\omega, x) - F_{T_{j_k}}(x)| \to 0$ for all $\omega \in \Omega_4$. The result then follows by using the equivalence (in the other direction) of Theorem 6.3.1 of Resnick (1999).