

On limit theorems for persistent Betti numbers from dependent data*

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Abstract

We study persistent Betti numbers and persistence diagrams obtained from a time series and random fields. It is well known that the persistent Betti function is an efficient descriptor of the topology of a point cloud. So far, convergence results for the (r, s) -persistent Betti number of the q th homology group, $\beta_q^{r,s}$, were mainly considered for finite-dimensional point cloud data obtained from i.i.d. observations or a Poisson sampling scheme. In this article, we extend these considerations. We derive limit theorems for the pointwise convergence of persistent Betti numbers $\beta_q^{r,s}$ in quite general dependence settings.

Keywords: Dependent data; Functional data; Limit theorems; Markov Chains; Marton Coupling; Persistent Betti numbers; Persistence diagrams; Point processes; Time series; Topological data analysis; Random fields; Random geometric complexes.

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Topological data analysis (TDA) is a comparably young field in applied mathematics at the intersection between computational geometry, probability theory, mathematical statistics and machine learning. An introduction offer the monograph of Chazal and Michel [2017] and the survey of Wasserman [2018]. In this article, we will focus on a topic in persistent homology, which is major branch in TDA: We study the large sample behavior of persistent Betti numbers and the corresponding persistence diagram obtained from time series or random fields.

So far, the literature has focused on point cloud data obtained from two major sources. On the one hand, there are various limit theorems for persistent Betti numbers obtained from stationary point processes as a quite general class. On the other hand, the binomial process, i.e., a sample of i.i.d. data, and its counterpart the Poisson process are also intensely studied.

In an early contribution, Kahle [2011] investigates the asymptotic behavior of the expectation of Betti numbers in the sub-, supercritical and critical regime. An extension of these results, give Yogeshwaran and Adler [2015]. From the above mentioned three asymptotic regimes, the critical (or thermodynamic) regime certainly gets the most attention and in the remainder of the introduction, we will limit the discussion to this case.

One of the first major contributions which studies large deviation inequalities and central limit theorems for the Poisson and binomial sampling scheme in the critical regime is the work of Yogeshwaran et al. [2017]. Extensions to persistent Betti numbers and persistence diagrams are given in Hiraoka et al. [2018]. Krebs and Polonik [2019] establish the strong stabilizing property in the sense of Penrose and Yukich for persistent Betti numbers. Strong laws of large

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numbers for Betti numbers obtained from the Poisson or the binomial process on general manifolds are considered in Goel et al. [2018]. Other recent contributions which also discuss limiting results for Betti numbers are Owada [2018], Trinh [2018], Owada and Thomas [2018]; the convergence of persistence diagrams is also studied in Divol and Polonik [2018].

In the context of time series, the behavior of Betti numbers has been mainly investigated in applications. Classification problems for time series using methods from TDA are considered in Seversky et al. [2016] and in Umeda [2017]. The applications of TDA to networks obtained from financial data are studied in Gidea [2017] and Gidea and Katz [2018]; here the methods of TDA measure a type of high-dimensional and time-dependent correlation in the network.

The aim of this paper, is to provide two advances in the study of persistent Betti numbers in the context of time series and random fields.

On the one hand, we study the large sample behavior of the expectation of persistent Betti numbers obtained from a stationary time series or a random field. More precisely, for the time series case, let $X = (X_t : t \in \mathbb{Z}) \subseteq [0, 1]^p$ be a stationary Markov chain of order m (w.r.t. its natural filtration) with a continuous and strictly positive joint density g of (X_1, \dots, X_{m+1}) . Write κ for the marginal density of each X_t . It is well-known that for a binomial process X^* which consists of i.i.d. observations X_t^* with marginal density κ the limit of $n^{-1} \mathbb{E} [\beta_q^{r,s}(\mathcal{K}((n^{1/p} \mathbb{X}_n^*)))]$ exists, here we write \mathcal{K} for the filtration (see below). Using the nearly additive properties of persistent Betti numbers, we show that Markov chains converge to the same limit, in fact,

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E} [\beta_q^{r,s}(\mathcal{K}(n^{1/p} \mathbb{X}_n))] = \mathbb{E} [\hat{b}_q(\kappa(X_t)^{1/p}(r, s))], \quad \forall q \in \{0, \dots, p-1\}, \quad \forall 0 \leq r \leq s < \infty,$$

and where $\hat{b}_q(r, s)$ is the limit of $n^{-1} \mathbb{E} [\beta_q^{r,s}(\mathcal{K}((n^{1/p} \mathbb{Y}_n^*)))]$ for a homogeneous binomial process \mathbb{Y}_n^* on $[0, 1]^p$ with $\kappa \equiv 1$. We also prove a related strong law of large numbers. Doing so, we can also conclude convergence results for persistence diagrams. Moreover, we establish similar convergence results for stationary random fields.

On the other hand, we establish an exponential inequality and give strong laws of large numbers for persistent Betti numbers, which are not exclusively derived from point clouds on \mathbb{R}^p . Instead, we also allow for functional data as a potential data source. The presented exponential inequality relies on the concept of the Marton coupling, see Marton [2003]. Marton couplings have also been successfully used in the past to derive concentration inequalities of the McDiarmid-type, see also Samson [2000] and Paulin [2015].

The remainder of this paper is organized as follows. In Section 1, we give the notation used throughout the manuscript. Furthermore, we outline the basic concept of persistent homology. In Section 2, we describe the dependence structure assumed for our time series model and present our main results related to the time series case. In Section 3, we study the extension of our results to random fields. The proofs are contained in Section 4; further deferred calculations are contained in the Appendix A.

1 Notation and assumptions

The aim of this section is not to make the paper self-contained which is impossible. The aim is rather to allow the reader from other areas to become familiar with the vocabulary and to understand the basic concepts of topological data analysis. For some background reading, we refer the reader to Wasserman [2018] and Chazal and Michel [2017].

We begin with some general notation. We write $\#A$ for the cardinality of a countable set A . Let (S, \mathfrak{S}, μ) be a measure space and let d be a metric on S . Then we write $B(x, r) = \{y \in S : d(x, y) \leq r\}$ for the closed d -ball around x . The diameter of a set $A \subseteq S$ is $\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}$. Write $\otimes_{i=1}^{\ell} \mu = \mu^{\otimes \ell}$ for the ℓ -fold product measure of a measure μ on the product space $(S^{\ell}, \mathfrak{S}^{\otimes \ell})$. The essential supremum of a real-valued function f defined on (S, \mathfrak{S}, μ) is abbreviated by $\|f\|_{\infty, \mu}$. We write simply $\|f\|_{\infty}$ for the supremum norm of a continuous function on \mathbb{R}^p .

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(S, \mathfrak{S}), (T, \mathfrak{T})$ be two state spaces. Consider two random variables $X : \Omega \rightarrow S$ and $Y : \Omega \rightarrow T$. Assume that X admits a regular conditional distribution given Y . We write $\mathbb{M}_{X|Y} : T \times$

$\mathfrak{S} \rightarrow [0, 1]$ for this distribution.

We construct the filtration from the Čech (\mathcal{C}) or the Vietoris-Rips (\mathcal{R}) complex. If \mathbb{X} is a finite point cloud on the measure space (S, \mathfrak{S}, μ) equipped with the metric d and $r \geq 0$, these complexes are defined by

$$\begin{aligned}\mathcal{C}(\mathbb{X}, r) &= \left\{ \text{finite } \sigma \subset \mathbb{X}, \bigcap_{x \in \sigma} B(x, r) \neq \emptyset \right\} \text{ and} \\ \mathcal{R}(\mathbb{X}, r) &= \left\{ \text{finite } \sigma \subset \mathbb{X}, \text{diam}(\sigma) \leq r \right\}.\end{aligned}$$

In the following, the writing $\mathcal{K}(\mathbb{X}, r)$ refers to both the Čech and the Vietoris-Rips complex. The corresponding filtration is given by $\mathcal{K}(\mathbb{X}) = \{\mathcal{K}(\mathbb{X}, r) : 0 \leq r < \infty\}$.

In this manuscript, we take a shortcut when it comes to explaining the background on persistent homology, which then yields the definition of the persistent Betti numbers. We shortly define the persistence diagram and then deduce a geometric definition of the persistent Betti numbers from it.

We use the field \mathbb{F}_2 to build homology groups $H_q(\mathcal{K}(\mathbb{X}, r))$ for $q \geq 0$. The q th persistence diagram summarizes the evolution of the q th homology group; it is a multiset of points in $\Delta = \{(b, d) : 0 \leq b < d \leq \infty\}$. So each point (b, d) in the q th persistence diagram corresponds to a q -dimensional hole (feature) in the filtration $\mathcal{K}(\mathbb{X})$ which is born (appears for the first time) at time b and dies (disappears in the filtration) at time d . The lifetime of this feature is $d - b$ and is called the persistence of (b, d) . $d = \infty$ means that the feature has an infinite lifetime, i.e., it lives until infinity.

Persistence diagrams exist given mild assumptions on the filtration, see Chazal et al. [2016]. Also in the case of a random point cloud, e.g., an i.i.d. sample, the persistence diagram can inherit certain smoothness properties from the point cloud, see Chazal and Divol [2018].

Let $\mathfrak{D}_q(\mathbb{X}) = \{(b_i, d_i) \in \Delta : i = 1, \dots, n_q\}$ be the q th persistence diagram given as a multiset of points. Then in the following we understand $\mathfrak{D}_q(\mathbb{X})$ as a counting measure on Δ defined as

$$\xi_q(\mathbb{X}) = \sum_{(b_i, d_i) \in \mathfrak{D}_q(\mathbb{X})} \delta_{(b_i, d_i)}.$$

Moreover, for a pair (r, s) , $0 \leq r \leq s$, the q th persistent Betti number with parameter (r, s) is defined as

$$\begin{aligned}\beta_q^{r,s}(\mathcal{K}(\mathbb{X})) &= \xi_q(\mathbb{X})([0, r] \times (s, \infty]) \\ &= \#\{(b_i, d_i) \in \mathfrak{D}_q(\mathbb{X}) : 0 \leq b_i \leq r \leq s < d_i\}.\end{aligned}$$

This means that $\beta_q^{r,s}$ counts the number of q -dimensional features born before time r and still alive at time s , i.e., the q -dimensional features in the upper left rectangular area with vertex (r, s) . So given $r \ll s$, the persistent Betti number $\beta_q^{r,s}$ represents the number of q -dimensional features with a high persistence. It is clear that the values of the persistent Betti function $\{\beta_q^{r,s}(\mathcal{K}(\mathbb{X})) : 0 \leq r \leq s\}$ also describe the q th persistence diagram $\xi_q(\mathbb{X})$ completely.

Given a metric space (E, d) and Radon measures ν, ν_1, ν_2, \dots , we say that $(\nu_n)_{n \in \mathbb{N}_+}$ converges vaguely to ν if

$$\int_E f d\nu_n \rightarrow \int_E f d\nu, \quad \forall f \in C_c(E),$$

where $C_c(E)$ is the class of all continuous functions on E with compact support. We indicate this by writing $\nu_n \xrightarrow{v} \nu$.

Given a time series $X_1^n = (X_1, \dots, X_n) \subseteq S$, we write $\mathbb{X}_n = \{X_1, \dots, X_n\}$ for the associated point cloud which has no ordering. We also write x_1^n for (x_1, \dots, x_n) . The dimension of a simplex $\sigma \in \mathcal{K}$ is its cardinality minus 1. If σ has dimension j , we call σ a j -simplex. Write $\mathcal{K}_j(\mathbb{X}, r)$ for the set of j -simplices in $\mathcal{K}(\mathbb{X}, r)$. Moreover, for a measurable set $A \in \mathfrak{S}$, we write $\mathcal{K}_j(\mathbb{X}, r; A)$ for the number of j -simplices in $\mathcal{K}(\mathbb{X}, r)$ with at least one vertex in A .

2 Persistent Betti numbers obtained from time series

This section contains the main results of this paper. We derive an exponential inequality for persistent Betti numbers from a quite general class of stochastic processes, which also applies to functional data and random fields. For the special case of an \mathbb{R}^p -valued time series, we also give the large sample behavior of the expectation and study the vague convergence of the corresponding persistence diagram.

The data generating process. Consider a stationary process $X = (X_t : t \in \mathbb{Z})$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and taking values in a Polish space S which is part of a measure space (S, \mathfrak{S}, μ) , where the measure μ is σ -finite.

The observations X_t admit a density κ w.r.t. μ . Furthermore, the observations admit certain conditional densities. More precisely, $\mathcal{L}(X_t | X_1, \dots, X_{t-1})$ admits a density $f_{X_t | X_1, \dots, X_{t-1}}$ for each $t \in \mathbb{N}$. Also $\mathcal{L}(X_{v_1}, \dots, X_{v_\ell} | X_t)$ admits a density $f_{X_{v_1}, \dots, X_{v_\ell} | X_t}$ for all $t, \ell \in \mathbb{N}_+$ and sets $\{v_1, \dots, v_\ell\} \subseteq \mathbb{N}_+$ of pairwise different indices which do not contain t . Moreover, there are $0 < f_* \leq f^* < \infty$ such that uniformly

$$f_* \leq \kappa \leq f^*, \quad f_{X_t | X_1, \dots, X_{t-1}} \leq f^* \quad \text{and} \quad f_{X_{v_1}, \dots, X_{v_\ell} | X_t} \leq f^* \quad (2.1)$$

for all $t, \ell \in \mathbb{N}_+$ and sets $\{v_1, \dots, v_\ell\} \subseteq \mathbb{N}_+$ of pairwise different indices which do not contain t . These requirements are not restrictive and satisfied for a wide range of smooth stochastic processes.

We use the concept of Marton couplings for the derivation of the main results. These couplings were first defined in Marton [2003] and measure the strength of dependence within a collection of random variables by a mixing (or coupling) matrix.

Definition 2.1 (Marton coupling). Let $N \in \mathbb{N}$ and let $\Lambda_1, \dots, \Lambda_N$ be Polish. Let $Z = (Z_1, \dots, Z_N)$ be a vector of random variables taking values in $\Lambda = \Lambda_1 \times \dots \times \Lambda_N$. A Marton coupling of Z is a set of couplings

$$(Z^{(z_1, \dots, z_i, z'_i)}, Z'^{(z_1, \dots, z_i, z'_i)}), \text{ for every } i = 1, \dots, N \text{ and every } z_1 \in \Lambda_1, \dots, z_i, z'_i \in \Lambda_i$$

which satisfies the conditions

- (i) $Z_j^{(z_1, \dots, z_i, z'_i)} = z_j$ for all $j = 1, \dots, i$,
 $Z'_j{}^{(z_1, \dots, z_i, z'_i)} = z_j$ for all $j = 1, \dots, i-1$ and $Z'_i{}^{(z_1, \dots, z_i, z'_i)} = z'_i$.
- (ii) $(Z_{i+1}^{(z_1, \dots, z_i, z'_i)}, \dots, Z_N^{(z_1, \dots, z_i, z'_i)}) \sim \mathcal{L}(Z_{i+1}, \dots, Z_N | Z_1 = z_1, \dots, Z_{i-1} = z_{i-1}, Z_i = z_i)$,
 $(Z'_{i+1}{}^{(z_1, \dots, z_i, z'_i)}, \dots, Z'_N{}^{(z_1, \dots, z_i, z'_i)}) \sim \mathcal{L}(Z_{i+1}, \dots, Z_N | Z_1 = z_1, \dots, Z_{i-1} = z_{i-1}, Z_i = z'_i)$.
- (iii) If $z_i = z'_i$, then $Z^{(z_1, \dots, z_i, z'_i)} = Z'^{(z_1, \dots, z_i, z'_i)}$.

Write $\mathbb{M}_{Z_i | (Z_1, \dots, Z_{i-1})}$ for the conditional distribution of Z_i given (Z_1, \dots, Z_{i-1}) for $1 \leq i \leq N$. Construct a measure μ_i on the product space $\Lambda_1 \times \dots \times \Lambda_{i-1} \times \Lambda_i \times \Lambda_i$, which consists of the joint distribution of (Z_1, \dots, Z_{i-1}) and the product measure $\mathbb{M}_{Z_i | (Z_1, \dots, Z_{i-1})} \otimes \mathbb{M}_{Z_i | (Z_1, \dots, Z_{i-1})}$:

$$\begin{aligned} \mu_i(A) = & \int_{\Lambda_1 \times \dots \times \Lambda_{i-1}} \mathbb{P}_{(Z_1, \dots, Z_{i-1})}(\mathrm{d}(z_1, \dots, z_{i-1})) \int_{\Lambda_i} \mathbb{M}_{Z_i | (Z_1, \dots, Z_{i-1})}((z_1, \dots, z_{i-1}), \mathrm{d}z_i) \\ & \int_{\Lambda_i} \mathbb{M}_{Z_i | (Z_1, \dots, Z_{i-1})}((z_1, \dots, z_{i-1}), \mathrm{d}z'_i) \mathbb{1}\{A\}. \end{aligned} \quad (2.2)$$

Then, we define for a Marton coupling of Z the mixing matrix $\Gamma := (\Gamma_{i,j})_{1 \leq i, j \leq N}$ as an upper diagonal matrix with

$\Gamma_{i,i} = 1$ and

$$\Gamma_{j,i} = 0, \quad \Gamma_{i,j} = \operatorname{ess\,sup}_{\text{w.r.t. } \mu_i} \mathbb{P} \left(Z_j^{(z_1, \dots, z_{i-1}, z_i, z'_i)} \neq Z'_j{}^{(z_1, \dots, z_{i-1}, z_i, z'_i)} \right), \quad 1 \leq i < j \leq N,$$

where we compute the essential supremum w.r.t. the measure μ_i . Note that for $1 \leq i < j \leq N$ each entry in the mixing matrix is bounded above by

$$\Gamma_{i,j} \leq \sup_{(z_1, \dots, z_{i-1}, z_i, z'_i) \in \Lambda_1 \times \dots \times \Lambda_i \times \Lambda_i} \mathbb{P} \left(Z_j^{(z_1, \dots, z_{i-1}, z_i, z'_i)} \neq Z'_j{}^{(z_1, \dots, z_{i-1}, z_i, z'_i)} \right).$$

We return to the data generating process X . Write $\Gamma^{(n)}$ for the mixing matrix of the sample X_1, \dots, X_n . As X is stationary, $\Gamma_{i,j}^{(n)} = \Gamma_{i+k, j+k}^{(n)}$ (as long as all indices are between 1 and n). Consequently, $\Gamma_{i,j}^{(n)} = \Gamma_{n-j+1, n-i+1}^{(n)}$ for the choice $k = n - j - i + 1$. So the summation over all elements in line i is equivalent to the summation over all elements in column $n - i + 1$ (and vice versa), viz.,

$$\sum_{j=1}^n \Gamma_{i,j}^{(n)} = \sum_{j=i}^n \Gamma_{i,j}^{(n)} = \sum_{j=1}^{n-i+1} \Gamma_{j, n-i+1}^{(n)} = \sum_{j=1}^n \Gamma_{j, n-i+1}^{(n)}.$$

In particular, the maximum absolute column sum $\|\Gamma^{(n)}\|_1$ equals the maximum absolute row sum $\|\Gamma^{(n)}\|_\infty$. In what follows, we assume that the maximum absolute row sum of $\Gamma^{(n)}$ is uniformly bounded over all n , viz.,

$$\sup_{n \in \mathbb{N}} \|\Gamma^{(n)}\|_\infty < \infty. \quad (2.3)$$

Consider the spectral norm $\|\Gamma^{(n)}\|$ of the mixing matrix X . Using $\|\Gamma^{(n)}\|^2 \leq \|\Gamma^{(n)}\|_1 \|\Gamma^{(n)}\|_\infty$, we see that also $\|\Gamma^{(n)}\|$ is uniformly bounded in the spectral norm over all sample sizes, i.e., $\sup_{n \in \mathbb{N}} \|\Gamma^{(n)}\| < \infty$.

The condition on the mixing matrix in (2.3) is satisfied for a wide range of stochastic processes. Consider for instance, so-called delay embeddings for time series.

Example 2.2 (Delay embeddings from Markov chains). Let Z be a stationary, uniformly geometrically ergodic Markov chain in a Polish space \mathcal{E} whose marginal distribution and transition kernel both admit a density w.r.t. a reference measure μ . Construct a process X from Z via a delay embedding, that is, $X_t = (Z_t, Z_{t-\tau_1}, \dots, Z_{t-\tau_{m-1}}) \in \mathcal{E}^m$, where $\tau_1 < \dots < \tau_{m-1}$ are natural numbers. We show that this process X satisfies (2.3). We construct a Marton coupling $(X^{(x_1, \dots, x_i, x'_i)}, X'^{(x_1, \dots, x_i, x'_i)})$, for every $i = 1, \dots, n$ and every $x_1, \dots, x_{i-1}, x_i, x'_i \in \mathcal{E}^m$ with Goldstein's maximal coupling (Proposition A.3).

For every i and all states, Goldstein's maximal coupling yields a coupling $(X^{(x_1, \dots, x_i, x'_i)}, X'^{(x_1, \dots, x_i, x'_i)})$ such that (i), (ii) and (iii) from Definition 2.1 are satisfied. Moreover, the marginals of each coupling satisfy

$$\begin{aligned} & \mathbb{P} \left(X_j^{(x_1, \dots, x_i, x'_i)} \neq X'_j{}^{(x_1, \dots, x_i, x'_i)} \right) \\ & \leq d_{TV} \left(\mathcal{L}((X^{(x_1, \dots, x_i, x'_i)})_j^n), \mathcal{L}((X'^{(x_1, \dots, x_i, x'_i)})_j^n) \right) \\ & = d_{TV}(\mathcal{L}(X_j^n | X_1^{i-1} = x_1^{i-1}, X_i = x_i), \mathcal{L}(X'_j^n | X_1^{i-1} = x_1^{i-1}, X_i = x'_i)). \end{aligned} \quad (2.4)$$

Note that the left-hand side equals the coefficient $\Gamma_{i,j}^{(n)}$. Thus, we can easily bound above the norm of the mixing matrix $\Gamma^{(n)}$ with the properties of the Markov chain Z . For simplicity, we use $\Gamma_{i,j}^{(n)} \leq 1$ for $0 \leq j - i \leq \tau_{m-1}$ and only consider

the asymptotic properties for $j - i > \tau_{m-1}$. In that case, we can derive from the Markov property of Z

$$\mathbb{P}(X_j^n \in A | X_1^i = x_1^i) = \mathbb{P} \left\{ \left(\begin{pmatrix} Z_j \\ Z_{j-\tau_1} \\ \vdots \\ Z_{j-\tau_{m-1}} \end{pmatrix}, \dots, \begin{pmatrix} Z_n \\ Z_{n-\tau_1} \\ \vdots \\ Z_{n-\tau_{m-1}} \end{pmatrix} \right) \in A \mid Z_i = z_i \right\},$$

where $x_i = (z_i, \dots, z_{i-\tau_{m-1}})'$. Next, we use that the total variation distance of a Markov chain is determined by the observation closest to i . Consequently, if $j - i > \tau_{m-1}$, (2.4) equals

$$d_{TV}(\mathcal{L}(Z_{j-\tau_{m-1}} | Z_i = z_i), \mathcal{L}(Z_{j-\tau_{m-1}} | Z_i = z_i')).$$

By assumption, Z is uniformly geometrically ergodic, so this last quantity is at most $1 \wedge (C\varepsilon^{j-\tau_{m-1}-i})$ for a certain $\varepsilon \in (0, 1)$ and $C \in \mathbb{R}_+$. In particular, we have for a row of the mixing matrix of X_1, \dots, X_n

$$\Gamma_{i,\cdot}^{(n)} \leq (1, \dots, 1, 1 \wedge (C\varepsilon), 1 \wedge (C\varepsilon^2), \dots, 1 \wedge (C\varepsilon^{n-1-\tau_{m-1}})).$$

In particular, (2.3) is satisfied.

As we consider general state spaces, we also need a covering condition which is satisfied in many examples.

Condition 2.3 (Covering condition). *The state space (S, \mathfrak{S}) is precompact. Write $N = N(r, S, d)$ for the r -covering number of S w.r.t. d , i.e., for each $r > 0$, S admits a covering $\{B(w_j, r) : 1 \leq j \leq N\}$ with balls w.r.t. the metric d of diameter r located at positions w_j .*

Let I be an interval which may depend on the space (S, \mathfrak{S}) or may be unbounded. There is a sequence of scaling factors $(\eta_n : n \in \mathbb{N}) \subseteq \mathbb{R}_+$ such that for each $r \in I \subseteq \mathbb{R}$ there is an $\alpha \in \mathbb{R}$ such that

$$\limsup_{n \rightarrow \infty} n \sup_{w \in S} \mu(B(w, 4\eta_n^{-1}r)) (\log n)^{-\alpha} = \mathcal{O}(1). \quad (2.5)$$

Also the logarithm of the $(4\eta_n^{-1}r)$ -covering number of S w.r.t. the metric d grows at most polynomially in $\log n$, i.e.,

$$\log N(4\eta_n^{-1}r, S, d) = \mathcal{O}((\log n)^{\bar{\alpha}}), \text{ for some } \bar{\alpha} \in \mathbb{R}_+. \quad (2.6)$$

Some discussion on the covering condition is needed. (2.6) is clearly needed to bound above the complexity of the underlying metric space (S, d) . Condition (2.5) is more delicate as it regulates the ratio between the number of points n and the μ -volume of the d -ball. Many spaces satisfy this condition. We give here two examples, the finite-dimensional case, i.e., \mathbb{R}^p , and the functional case.

Example 2.4 (Coverings for finite-dimensional spaces). Consider the classical case $S = [0, 1]^p$ which is endowed with the ∞ -norm and the Lebesgue measure $|\cdot|$. For each $r > 0$, $[0, 1]^p$ can be covered with disjoint cubes of side length at most r , i.e., balls w.r.t. the ∞ -norm $B(w_j, r) = w_j + [-2^{-1}r, 2^{-1}r]^p$. In that case, one finds with geometric arguments that any ball of diameter r at position w can be covered with at most 2^p balls of diameter r at fixed positions w_j .

In this Euclidean setting, three regimes are classically studied. In the subcritical regime $\eta_n^{-1}n^{1/p} \rightarrow 0$, i.e., the scaling factors grow faster than $n^{1/p}$. In the critical regime, this growth is balanced, so that $\eta_n^{-1}n^{1/p} \rightarrow \eta^{-1}$. Moreover, $\eta_n^{-1}n^{1/p} \rightarrow \infty$ in the supercritical regime.

We study the situation for a point cloud of n points $\mathbb{X}_n \subseteq [0, 1]^p$ obtained from a stationary time series X_1, \dots, X_n whose marginals admit a density κ w.r.t. the Lebesgue measure. In the subcritical regime, the points of the rescaled point cloud $\eta_n \mathbb{X}_n$ tend to become more and more isolated as the number of points per volume tends to zero. In the critical

regime, the number of points per volume from $\eta_n \mathbb{X}_n$ tends to a constant. In the supercritical regime, the points from the point cloud $\eta_n \mathbb{X}_n$ lie increasingly dense.

Clearly, scaling factors such as $\eta_n = n^{1/p}$, which achieve the thermodynamic regime, satisfy the condition from (2.5), viz., $n|B(x, \eta_n^{-1}s)|$ is proportional to s^p . The covering number $N(\eta_n^{-1}s, [0, 1]^p, \|\cdot\|_\infty)$ is in this case proportional to n .

Moreover, note that we can also allow for a slower increase in η_n which then yields a supercritical regime. For instance, η_n proportional to $n^{1/p}(\log n)^{-\tilde{\alpha}}$ still satisfies (2.5) (and also (2.6)) for each $\tilde{\alpha} > 0$.

Scaling factors which achieve a subcritical regime clearly satisfy (2.5), however, (2.6) restricts the growth rate from above; for instance, any polynomial rate is allowed for (2.6), i.e., $\eta_n = n^\alpha$ for some $\alpha > 0$.

Example 2.5 (Coverings of functional spaces). Let $\alpha > 0$. We study the class of all functions x on the unit interval that posses uniformly bounded derivatives up to order $\underline{\alpha}$ (the greatest integer strictly smaller than α) and whose highest derivatives are Hölder continuous of order $\alpha - \underline{\alpha}$. Write $D^k x$ for the k th derivative of a function x . Define

$$\|x\|_\alpha = \max_{k \leq \underline{\alpha}} \sup_{t \in [0, 1]} |D^k x(t)| + \sup_{\substack{s, t \in (0, 1), \\ s \neq t}} \frac{|D^{\underline{\alpha}} x(t) - D^{\underline{\alpha}} x(s)|}{|s - t|^{\alpha - \underline{\alpha}}}.$$

Write $C_M^\alpha([0, 1])$ for the set of continuous real-valued functions on $[0, 1]$ with $\|x\|_\alpha \leq M$. We write $\|\cdot\|_\infty$ for the supremum-norm of a real-valued function on $[0, 1]$ and denote (in this example) the corresponding r -neighborhood of x by $B(x, r)$ (other norms are also possible). Then Theorem 2.7.1 in van der Vaart and Wellner [1996] yields that the ε -covering number of $C_1^\alpha([0, 1])$ w.r.t. the supremum-norm satisfies

$$\log N(\varepsilon, C_1^\alpha([0, 1]), \|\cdot\|_\infty) \leq c\varepsilon^{-1/\alpha},$$

for a certain constant $c \in \mathbb{R}_+$ which is independent of $\varepsilon > 0$. Also one has for the distribution μ of many stationary stochastic processes on $[0, 1]$ that

$$\mu(B(x, s)) \sim C_x s^{-\rho} \exp(-Cs^{-\beta}), \quad s \rightarrow 0, \quad (2.7)$$

for $\rho, \beta, C \geq 0, C_x > 0$, see also Mayer-Wolf and Zeitouni [1993] and Li and Shao [2001]. For instance, for a centered Gaussian measure, we have for any element x in the reproducing Hilbert space generated by μ

$$C_x^{-1} \mu(B(0, s)) \leq \mu(B(x, s)) \leq \mu(B(0, s)) = (4/\pi) \exp(-\pi^2/(8s^2)), \quad (2.8)$$

where $C_x > 0$, see Li and Shao [2001] Theorem 3.1. So in this case $\rho = 0$.

Assume that the parameter range I for the radii s is an interval $[s, \bar{s}] \subseteq \mathbb{R}_+$. Define the scaling factor by $\eta_n = (\log n)^{1/\beta} \bar{s} C^{-1/\beta}$. Then, given that $\sup_{x \in S} C_x < \infty$, (2.5) is

$$n \sup_{x \in S} \mu(B(x, \eta_n^{-1}s)) \sim \sup_{x \in S} C_x \left(\frac{\bar{s}}{s C^{1/\beta}} \right)^\rho (\log n)^{\rho/\beta} n^{1 - (\bar{s}/s)^\beta} = \mathcal{O}((\log n)^{\rho/\beta} n^{1 - (\bar{s}/s)^\beta}).$$

So for the Gaussian measure from (2.8) where $\rho = 0$ and in the case where $s = \bar{s}$, we can even achieve a functional thermodynamic regime in the sense that $n\mu(B(x, \eta_n^{-1}s)) \rightarrow a_x$ for some $a_x \in \mathbb{R}_+$ as $n \rightarrow \infty$. If $s \leq \bar{s}$, we have a subcritical regime in the sense that $n\mu(B(x, \eta_n^{-1}s)) \rightarrow 0$ as $n \rightarrow \infty$.

Now let $\rho \neq 0$. Then for the above choice of η_n , if $\bar{s} = s$ we have a supercritical regime in the sense that $n\mu(B(x, \eta_n^{-1}s))$ is of order $(\log n)^{\rho/\beta}$ as $n \rightarrow \infty$. If $s < \bar{s}$, we have again subcritical regime.

So the limiting regime depends crucially on the choice of \bar{s} in the definition of η_n in this example. Moreover, in this setting, it is impossible to obtain a critical regime for an entire range I of radii s . This is a major difference to the

finite-dimensional Euclidean setting.

Moreover, in the general setting from (2.7), the covering number $N(\eta_n^{-1}s, C_1^\alpha(\mathcal{D}), \|\cdot\|_\infty)$ satisfies (2.6) and is of order $\mathcal{O}((\log n)^{1/(\alpha\beta)}(\bar{s}s^{-1}C^{-1/\beta})^{1/\alpha})$.

We come to the first main result in this article. To this end, we assume that the stochastic process satisfies (2.1) and admits a Marton coupling which fulfills (2.3). Also, the state space satisfies Condition 2.3.

Theorem 2.6 (Concentration inequality for persistent Betti numbers). *For each $q \in \mathbb{N}$, for each (r, s) with $r \leq s$, for $a > 1/2$ and $\gamma = (2a - 1)/(1 + 2q)$, there are three constants $C_1, \dots, C_3 \in \mathbb{R}_+$ depending on r, s, q but neither on n nor on t such that*

$$\begin{aligned} & \mathbb{P}(|\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_n)) - \mathbb{E}[\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_n))]| \geq n^\alpha t) \\ & \leq 2 \exp(-C_1 n^{2a-1-2\gamma q} t^2) + C_2 (1 + n^{q-a} t^{-1}) n^{(1-\gamma)(q+1)} \exp(-n^\gamma + C_3 (\log n)^{\alpha\sqrt{\alpha}}). \end{aligned}$$

In particular, $n^{-1}|\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_n)) - \mathbb{E}[\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_n))]| \rightarrow 0$ a.s.

It remains to show that the normalized expectation of the persistent Betti numbers converges to a limit. Here we restrict our considerations to point cloud data on \mathbb{R}^p for the following reason: In order to obtain limit theorems for dependent point cloud data on a general measure space, one first has to derive the limit of $n^{-1}\mathbb{E}[\beta_q^{r,s}(\mathcal{K}(\eta_n \mathcal{P}_n))]$ for a certain underlying Poisson process \mathcal{P}_n on this measure space, see for instance Last and Penrose [2017] for the definition of a Poisson process on a general measure space. Then one needs to apply a de-Poissonization argument to obtain a limit for the binomial process which treats the situation for an i.i.d. sample. The nearly additive properties of the Betti numbers allow then to conclude the case also for time series data. This whole problem is quite comprehensive and so far – to the best of our knowledge – these extensions have not been considered in the literature, so we have decided to limit our considerations to \mathbb{R}^p -valued data.

We study two kind of processes $X = (X_t : t \in \mathbb{Z}) \subseteq [0, 1]^p$ in the critical regime, namely, (1) processes which can be coupled to a process with a discrete state space and (2) Markov chains of finite order.

First consider a process X which is obtained from a stationary discrete process $Z = (Z_t : t \in \mathbb{Z})$ as follows. Let κ be a blocked density w.r.t. the Lebesgue measure on $[0, 1]^p$, i.e., there is an $m \in \mathbb{N}_+$ such that

$$\kappa = \sum_{i=1}^{m^p} \alpha_i \mathbf{1}\{A_i\}, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{m^p} \alpha_i |A_i| = 1 \quad (2.9)$$

and where the subcubes A_i partition $[0, 1]^p$. Note that the A_i may have different volumes.

The process Z takes values in a finite set $S = (s_1, \dots, s_{m^d})$, $s_i \neq s_j$ for $1 \leq i \neq j \leq m^p$, such that $\mathbb{P}(Z_t = s_i) = \alpha_i |A_i|$. Also Z admits a Marton coupling which satisfies (2.3). Define X_t (conditional on Z_t) by

$$X_t = \sum_{i=1}^{m^p} Y_{t,i} \mathbf{1}\{Z_t = s_i\},$$

where the $Y_{t,i}$ are independent and uniformly distributed on A_i for $1 \leq t \leq n$ and $1 \leq i \leq m^d$. Then if $B \subseteq A_i$, $\mathbb{P}(X_t \in B) = (|B|/|A_i|)(\alpha_i |A_i|) = \alpha_i |B|$. Hence, each X_t admits a marginal density κ .

The conditional distribution of the process X works as follows. In the first step and conditional on the past (X_1, \dots, X_{t-1}) , we choose a subcube A_i , according to the weight function

$$\mathbb{P}(X_t \in A_i | X_1 \in A_{j_1}, \dots, X_{t-1} \in A_{j_{t-1}}) = \mathbb{P}(X_t \in A_i | Z_1 = s_{j_1}, \dots, Z_{t-1} = s_{j_{t-1}}).$$

In the second step, we choose at random a point in the subcube A_i as the realization of X_t .

Consequently, $X = (X_t : t \in \mathbb{N})$ admits a Marton coupling which satisfies (2.3). The conditional distribution of X is invariant in the sense that

$$\mathbb{P}(X_t^{t+\ell} \in A | X_{t-s}^{t-1} = x_{t-s}^{t-1}) = \mathbb{P}(X_t^{t+\ell} \in A | X_{t-s}^{t-1} = y_{t-s}^{t-1}), \quad (2.10)$$

for all $x_i, y_i \in A_i, i \in \{t-s, \dots, t-1\}, \ell, s \in \mathbb{N}, t-s \geq 1$. If we can only observe the process X , then we can think of Z as a hidden process. We have the following theorem.

Theorem 2.7. *Let $X = (X_t : t \in \mathbb{Z})$ be an $[0, 1]^p$ -valued process which admits a Marton coupling that satisfies (2.3). Each X_t has a marginal density κ as in (2.9) such that $0 < \inf \kappa \leq \sup \kappa < \infty$. Also the transition probabilities fulfill (2.10). Then for each $q \in \{0, \dots, p-1\}$*

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \mathbb{X}_n)) \right] = \mathbb{E} \left[\hat{b}_q(\kappa(X_t)^{1/p}(r, s)) \right], \quad (2.11)$$

$$\lim_{n \rightarrow \infty} n^{-1} \beta_q^{r,s}(\mathcal{K}(n^{1/p} \mathbb{X}_n)) = \mathbb{E} \left[\hat{b}_q(\kappa(X_t)^{1/p}(r, s)) \right] \quad a.s., \quad (2.12)$$

where $\hat{b}_q(r, s)$ is the limit of $n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \mathbb{X}_n^*)) \right]$ for a homogeneous binomial process \mathbb{X}_n^* on $[0, 1]^p$.

So the persistent Betti number obtained from this kind of time series converges to the same limit as the persistent Betti number of the corresponding binomial process both *a.s.* and in the mean.

We extend Theorem 2.7 to general marginal density functions $\kappa: [0, 1] \rightarrow \mathbb{R}_+$ which can be approximated by blocked density functions κ_ε . To this end, we restrict ourselves to the case of uniformly ergodic Markov chains of order m , viz., $X = (X_t : t \in \mathbb{Z})$ is a stationary process such that $\mathcal{L}(X_t | X_u : u < t) = \mathcal{L}(X_t | X_{t-1}, \dots, X_{t-m})$, for some $m \in \mathbb{N}_+$. For such a Markov chain all transition probabilities are determined by the joint density g of X_1, \dots, X_{m+1} which is assumed to be continuous and strictly positive on $[0, 1]^{(m+1)p}$ in that $\inf\{g(z) : z \in [0, 1]^{(m+1)p}\} > 0$.

It is known that this kind of aperiodic restriction ensures the Markov chain X to be uniformly geometrically ergodic, see also Meyn and Tweedie [2012] Theorem 16.0.2.

Furthermore, the limit on the right-hand side in (2.11) is continuous: Indeed, Divol and Polonik [2018] show that

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left[\hat{b}_q(\kappa_\varepsilon(Y_\varepsilon)^{1/p}(r, s)) \right] = \mathbb{E} \left[\hat{b}_q(\kappa(Y)^{1/p}(r, s)) \right], \quad (2.13)$$

where κ_ε are blocked density functions on $[0, 1]^p$ (from a regular grid) as in (2.9) which converge to κ in the $\|\cdot\|_\infty$ -norm and where the Y_ε (resp. Y) have density κ_ε (resp. κ).

For this kind of Markov chains X we obtain from the previous Theorem 2.7 the following result.

Theorem 2.8. *Let X be a homogeneous Markov chain of order m taking values in $[0, 1]^p$ such that the joint density g of X_1, \dots, X_{m+1} is continuous and satisfies $\inf\{g(z) : z \in [0, 1]^{(m+1)p}\} > 0$. The X_t have marginal density κ . Then for each $q \in \{0, \dots, p-1\}$ the convergence results from (2.11) and (2.12) in Theorem 2.7 are also valid.*

Consequently, we obtain also for this natural generalization of the binomial process the well-known limit. The generalization to arbitrary stationary processes X which admit a Marton coupling is rather elaborate and complex. Actually, when following the current scheme of the proof, one first has to assume that this process X can be coupled to a discrete process \tilde{X} which approximates X sufficiently closely in terms of the conditional distribution functions. This would mainly result in a complex notation. For this reason, we have limited our considerations to processes whose conditional distributions only depend on m lags of its past, this is sufficient for many applications and also serves as an approximation to the general case.

We conclude with an immediate result which follows from the Theorem 2.7 and Theorem 2.8 and the work of Hiraoka et al. [2018] concerning the vague convergence of persistence diagrams.

Corollary 2.9 (Vague convergence of persistent diagrams obtained from dependent data). *Let the assumptions from Theorem 2.7 or Theorem 2.8 be satisfied. Then for each $q \in \{0, \dots, p-1\}$, there is a Radon measure ξ_q depending on κ such that $\mathbb{E}[\xi_{n,q}] \xrightarrow{v_\zeta} \xi_q$ as $n \rightarrow \infty$. Also, $\xi_{n,q} \xrightarrow{v_\zeta} \xi_q$ a.s. as $n \rightarrow \infty$.*

3 Extensions to random fields

We extend our results from time series to random fields in two settings, these correspond then to the situations discussed in Theorems 2.7 and 2.8 for the time series case.

The extension to random fields requires mainly notational changes. As we consider stationary random fields indexed by the regular d -dimensional lattice \mathbb{Z}^d , the main difference is the ordering of the data which is typically located in the subset \mathbb{N}^d . If $u, v \in \mathbb{N}^d$ are two positions on the lattice, we write $u \geq v$ ($u \leq v$) if and only if $u_i \geq v_i$ ($u_i \leq v_i$) for all $i = 1, \dots, d$. Moreover, we construct a total ordering on \mathbb{N}^d with the ℓ^1 -sequence-norm as follows. Let $u, v \in \mathbb{N}^d$, then

$$u >_d v : \iff \exists j \in \{1, \dots, d\} : \|u_1^j\|_1 > \|v_1^j\|_1 \text{ and } \|u_1^k\|_1 = \|v_1^k\|_1 \text{ for } k = j+1, \dots, d, \quad (3.1)$$

where $\|u\|_1 = |u_1| + \dots + |u_d|$. The relations $>_d, \leq_d, \geq_d$ follow in the same spirit.

For a vector $N = (N_1, \dots, N_d) \in \mathbb{N}_+^d$, we denote the cardinality of the corresponding d -cube $\prod_{i=1}^d \{1, \dots, N_i\}$ by $\pi(N)$. For a given a random field $X = (X_u : u \in \mathbb{Z}^d)$ and an $N \in \mathbb{N}^d$, we write \mathbb{X}_N for the associated point cloud $\{X_u : u \leq N\}$, which represents the sample data. In the following, we will consider only such $N \in \mathbb{N}^d$ which satisfy

$$\min\{N_i : i = 1, \dots, d\} / \max\{N_i : i = 1, \dots, d\} \geq \bar{c},$$

for some constant $\bar{c} \in \mathbb{R}_+$. We write $N \rightarrow \infty$ for a sequence $(N(k) : k \in \mathbb{N}_+) \subseteq \mathbb{N}^d$ which satisfies this last relation for each $N(k)$ and also fulfills $\max\{N_i(k) : i = 1, \dots, d\} \rightarrow \infty$ as $k \rightarrow \infty$.

Consider a Marton coupling $((X_u, X'_u) : u \in \mathbb{N}_+^d)$ of a stationary random field on \mathbb{N}_+^d . We define the mixing matrix $\Gamma^{(\infty)}$ w.r.t. the ordering $>_d$. The line corresponding to location u in the mixing matrix $\Gamma^{(\infty)}$ is given by

$$\Gamma_{u,v}^{(\infty)} = \text{ess sup}_{\text{w.r.t. } \mu_u} \mathbb{P} \left(X_v^{(x_w : w <_d u, x_u, x'_u)} \neq X'_v^{(x_w : w <_d u, x_u, x'_u)} \right), \quad (3.2)$$

where $v \geq_d u$ and where μ_u is defined in the same spirit as μ_i in (2.2).

We study the entries of the mixing matrix in a simple example. Consider a stationary random field X on the lattice \mathbb{Z}^2 whose joint distribution can entirely be described by four (conditional) densities $f_{(0,0)} = \kappa, f_{(0,1)}, f_{(1,0)}$ and $f_{(1,1)}$. This means for any $N \in \mathbb{N}^d$ the joint distribution $\{X_u : u \leq N\}$ can be simulated with these four conditional densities and we can do this also using the ordering $<_d$, beginning at the corner point (1,1). So we first simulate $X_{(1,1)}$ according to $\kappa = f_{(0,0)}$. All observations $X_{(1,t)}$ for $1 < t \leq N_1$ (resp. $X_{(t,1)}$ for $1 < t \leq N_2$) are simulated with $f_{(0,1)}$ (resp. $f_{(1,0)}$). All remaining observations are simulated with the conditional density $f_{(1,1)}$. Figure 1 shows this situation.

Consider a location u in the lattice and a configuration of the Marton coupling which agrees at all locations of the past of u w.r.t. $>_d$. Consider a point v in the future of u w.r.t. $>_d$. Then the distributions of $X_v^{(x_w : w <_d u, x_u, x'_u)}$ and $X'_v^{(x_w : w <_d u, x_u, x'_u)}$ are affected by the different configurations at location u if and only if $v \geq u$. Hence, (3.2) is only affected by the locations v which satisfy $v \geq u$, which is a strict subset of all those locations v which satisfy $v \geq_d u$.

We come to the description of the dependence patterns. First we consider again the blocked density function from (2.9) and proceed as in the case for time series. Let $Z = (Z_u : u \in \mathbb{Z}^d)$ be a stationary random field on the regular d -dimensional lattice. The state space of Z is discrete, i.e., $S = \{s_1, \dots, s_{m^p}\}, s_i \neq s_j, \text{ for } 1 \leq i \neq j \leq m^p$, such that $\mathbb{P}(Z_u = s_i) = \alpha_i |A_i|$. Also Z admits a Marton coupling whose mixing matrix $\Gamma^{(\infty)}$ satisfies (similar as in (2.3))

$$\|\Gamma^{(\infty)}\|_\infty < \infty. \quad (3.3)$$

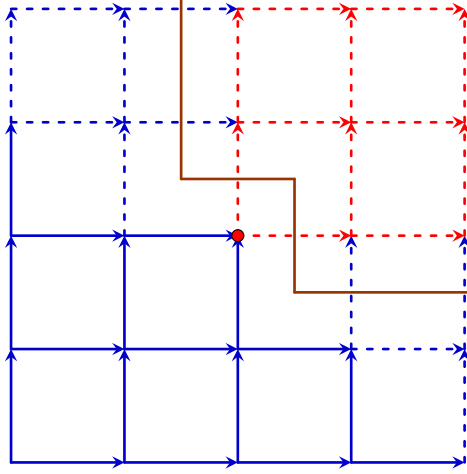


Figure 1: Factorization scheme of the conditional distribution functions of a stationary Markov random field on \mathbb{Z}^2 which can be described with the conditional densities $f_{(0,0)}, f_{(1,0)}, f_{(0,1)}$ and $f_{(1,1)}$. The arrows indicate the direction of the evolution (simulation) of the field. Solid blue arrows point to these locations which lie in the past (w.r.t. $>_d$) of the red dot (and in the present, in case of the red dot itself). The dashed arrows (red and blue) point at locations which lie in the future (w.r.t. $>_d$) of the red dot. The location with the red dot marks a point u , where the Marton coupling (X, X') disagrees conditional that it agrees on all locations from the past of u . The brown line separates the area, which is affected from the different states at location u , the corresponding arrows are red. Locations, which lie in the future, but are unaffected by the different states at location u , are marked in blue.

Define a new random field $X = (X_u : u \in \mathbb{Z}^d)$ (conditional on Z) by

$$X_u = \sum_{i=1}^{m^p} Y_{u,i} \mathbf{1}\{Z_u = s_i\},$$

where the $Y_{u,i}$ are independent and uniformly distributed on A_i for $u \in \mathbb{Z}^d$ and $1 \leq i \leq m^d$. Then if $B \subseteq A_i$, $\mathbb{P}(X_u \in B) = (|B|/|A_i|) (\alpha_i|A_i|) = \alpha_i|B|$. In particular, each X_u has a density κ on $[0, 1]^p$. Also all other properties from the time series case are inherited. Hence, we have once more an invariance property as in (2.10): For $N \in \mathbb{N}^d$ and $u, z_1, z_2 \leq N$ such that $z_2 <_d u <_d z_1$

$$\begin{aligned} & \mathbb{P}((X_v : u \leq_d v \leq_d z_1, v \leq N) \in A \mid X_w = x_w : z_2 \leq_d w <_d u, w \leq N) \\ &= \mathbb{P}((X_v : u \leq_d v \leq_d z_1, v \leq N) \in A \mid X_w = y_w : z_2 \leq_d w <_d u, w \leq N), \end{aligned} \quad (3.4)$$

for all $x_w, y_w \in A_{i_w}$, $1 \leq i_w \leq m^p$, for all w such that $z_2 \leq_d w <_d u$ and $w \leq N$.

Consequently, we obtain the following generalized variant of Theorem 2.7.

Theorem 3.1. *Let $X = (X_u : u \in \mathbb{Z}^d)$ be a $[0, 1]^p$ -valued random field on \mathbb{Z}^d , which admits a Marton coupling that satisfies (3.3) w.r.t. $>_d$ for each $N \in \mathbb{N}^d$. Each X_u has a marginal density κ as in (2.9) such that $0 < \inf \kappa \leq \sup \kappa < \infty$. Also the transition probabilities fulfill (3.4). Then for each $0 \leq q \leq p - 1$*

$$\lim_{N \rightarrow \infty} \pi(N)^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(\pi(N)^{1/p} \mathbb{X}_N)) \right] = \mathbb{E} \left[\hat{b}_q(\kappa(X_u)^{1/p}(r, s)) \right], \quad (3.5)$$

$$\lim_{N \rightarrow \infty} \pi(N)^{-1} \beta_q^{r,s}(\mathcal{K}(\pi(N)^{1/p} \mathbb{X}_N)) = \mathbb{E} \left[\hat{b}_q(\kappa(X_u)^{1/p}(r, s)) \right] \quad a.s. \quad (3.6)$$

We refer to Chazottes et al. [2007], Külske [2003] who consider couplings for high-temperature Gibbs measures for the discrete random field Z whose components take the values in $\{-, +\}$. Given certain upper bounds on the dependence

within the random field, they obtain for the two state Gibbs model a coupling (Z, Z') which satisfies

$$\mathbb{P}(Z_v \neq Z'_v | Z_u = +, Z'_u = -, Z_w = Z'_w, \forall w <_d u) \leq e^{-C\|x-y\|_1}, \quad (3.7)$$

for a certain constant $C \in \mathbb{R}_+$. So the probability of an unsuccessful coupling decays exponentially fast in the ℓ^1 -distance on the lattice, which is for two nodes x, y the minimal number of edges between x and y w.r.t. the standard 2^d -neighborhood structure. In particular, the Marton coupling (Z, Z') satisfies (3.3).

For a generalization of Theorem 2.8 we need a decay assumption on the mixing matrices. In the case of a Markov chain of finite order, Theorem 16.0.2 in Meyn and Tweedie [2012] states that strictly positive and continuous conditional densities ensure uniform geometric ergodicity. So concerning the Marton coupling, we also obtain a mixing matrix whose line entries decay at an exponential rate.

For random fields the situation is far more complex. To this end, we restrict ourselves to Markov random fields X of order 1 w.r.t. the 2^d neighborhood structure of the regular lattice \mathbb{Z}^d whose joint distribution can be described with 2^d (conditional) density functions

$$f_s : [0, 1]^p \times [0, 1]^{\|s\|_1 p} \rightarrow (0, \infty), \quad s \in \{0, 1\}^d, \text{ where } \kappa = f_0 : [0, 1]^p \rightarrow (0, \infty). \quad (3.8)$$

is the marginal density. More precisely, the distribution can be modeled with a scheme as in Figure 1, however, on a d -dimensional lattice. The conditional density f_s describes the transition within the set $\{z \in \mathbb{Z}^d : z_j = 0 \text{ for } j \in J(s)\}$, where $J(s) = \{1 \leq j \leq d : s_j = 0\}$.

We give an example for a cube $\mathcal{C}_N = \{u \in \mathbb{N}_+^d : u \leq N\}$. First we can simulate the random variable $X_{(1, \dots, 1)}$ in the lower left corner according to $\kappa = f_0$. Let e_k be the standard basis elements of \mathbb{R}^d for $k = 1, \dots, d$, i.e., the vector whose k th entry is 1 and 0 otherwise. Then the conditional densities f_{e_k} describe the transition on the coordinate axes of the cube. Similarly, with the remaining functions f_s , $s \neq (1, \dots, 1)$, we can completely simulate the transition on the lower envelope of the cube, i.e., the locations which are zero in at least one coordinate. Finally, the conditional density $f_{(1, \dots, 1)}$ describes the transition to those locations u , which are nonzero each entry.

It is an important fact that due to the Markov structure we can factorize the distribution of the random field on \mathcal{C}_N with these conditional densities and use the ordering $>_d$ in the same time.

In contrast to the one-dimensional situation of a Markov chain, it is this time not enough that the conditional densities from (3.8) are strictly positive in order to ensure a successful Marton coupling. To this end, we assume that the dependence within $X = \{X_u : u \in \mathbb{Z}^d\}$ decays at a polynomial rate in the sense that

$$\sum_{k=0}^{\infty} k^{2(d-1)} \max_{v: \|v-u\|_{\max}=k} \Gamma_{u,v}^{(\infty)} < \infty, \quad (3.9)$$

where $\|u\|_{\max} = \max\{|u_i| : i = 1, \dots, d\}$ and where $\Gamma^{(\infty)}$ is the mixing matrix of the random field restricted to \mathbb{N}_+^d . Note that a uniform exponential decay as in (3.7) is obviously sufficient for (3.9). Also note that due to the factorization property of X from (3.8), the mixing matrix at position (u, v) is nontrivial if and only if $v \geq u$. Also due to stationarity, it is entirely determined by the entries $\Gamma_{(1, \dots, 1), v}^{(\infty)}$ $v \geq (1, \dots, 1)$. Using last condition on the decay, we conclude with a generalized convergence result on the mean convergence of persistent Betti numbers obtained from Markov random fields.

Theorem 3.2. *Let the stationary random field X be completely characterized by the (conditional) density functions f_s from (3.8) which are all continuous. Each f_s is strictly positive on $[0, 1]^p \times [0, 1]^{\|s\|_1 p}$ in that $\inf_{x,y} f_s(x|y) > 0$. Moreover, the mixing matrix of X satisfies (3.9). Then X satisfies the convergence results from (3.5) and (3.6).*

4 Technical results

Before we come to the proofs of the central results, we need some preparations.

Lemma 4.1 (Concentration inequality for bounded transition kernels). *Let $Z = (Z_i : i \in \mathbb{N}_+)$ be a stochastic process which takes values in the measure space (S, \mathfrak{S}, μ) . Moreover assume that the conditional distributions $\mathcal{L}(Z_i | Z_j = z_j, 1 \leq j < i)$ admit a conditional density f_i . These densities are uniformly bounded in that*

$$f^* := \sup_{i \in \mathbb{N}_+} \|f_i(\cdot)\|_{\mu \otimes \mu^{\otimes (i-1)}, \infty} < \infty.$$

Let $(B_n : n \in \mathbb{N}_+)$ be a sequence of measurable subsets of S such that $\limsup_{n \rightarrow \infty} n\mu(B_n)(\log n)^{-\alpha} \leq c^$, for certain $\alpha, c^* \in \mathbb{R}_+$. Then there are constants $c_1, c_2 \in \mathbb{R}_+$ such that for all $n \in \mathbb{N}_+$ and $t \in \mathbb{R}_+$*

$$\mathbb{P}\left(\sum_{i=1}^n \mathbb{1}\{Z_i \in B_n\} > t\right) \leq \exp(-t + (e-1)f^*n\mu(B_n)) \leq c_1 \exp(-t + c_2(\log n)^\alpha c^*). \quad (4.1)$$

In particular, let Z be an \mathbb{R}^p -valued homogenous Markov chain which admits uniformly bounded conditional densities. For each n , let B_n be the r_n -neighborhood of a point $x \in \mathbb{R}^p$ such that $nr_n^p \rightarrow c^$. Then (4.1) holds with $\alpha = 0$.*

Proof. First, we bound the Laplace transform of $\mathbb{1}\{Z_i \in B_n\}$ w.r.t. \mathcal{F}_{i-1} , where $\{\mathcal{F}_i\}_i$ is the natural filtration of the process Z . We have

$$\begin{aligned} \mathbb{E}[\exp(\mathbb{1}\{Z_i \in B_n\}) | \mathcal{F}_{i-1}] &\leq e\mathbb{P}(Z_i \in B_n | Z_1, \dots, Z_{i-1}) + [1 - \mathbb{P}(Z_i \in B_n | Z_1, \dots, Z_{i-1})] \\ &\leq 1 + (e-1)f^*\mu(B_n) \leq \exp((e-1)f^*\mu(B_n)). \end{aligned}$$

Thus, we obtain for the entire process $\mathbb{P}(\sum_{i=1}^n \mathbb{1}\{Z_i \in B_n\} > t) \leq \exp(-t + (e-1)f^*n\mu(B_n))$, using Markov's inequality. This finishes the proof, noting that $\limsup_{n \rightarrow \infty} n\mu(B_n)(\log n)^{-\alpha} \leq c^*$ by assumption. \square

The next lemma is a generalization of Lemma 3.1 in Yogeshwaran et al. [2017].

Lemma 4.2. *Let $j \in \mathbb{N}$ and $X = \{X_t : t \in \mathbb{Z}\}$ be a process which takes values in a measure space (S, \mathfrak{S}, μ) . Let $\{v_1, \dots, v_\ell\}$ be a set of $\ell \leq j$ distinct natural numbers. Assume that the distribution of the vector $(X_{v_1}, \dots, X_{v_\ell})$, when conditioned on another observation X_i , $i \notin \{v_1, \dots, v_\ell\}$, admits a density $f_{(X_{v_1}, \dots, X_{v_\ell}) | X_i}$ w.r.t. $\mu^{\otimes \ell}$. Assume that these densities are uniformly essentially bounded in that*

$$\sup_{i \in \mathbb{Z}} \sup_{\substack{(v_1, \dots, v_\ell) \in \mathbb{Z}^\ell \\ i \neq v_j \neq v_k}} \|f_{(X_{v_1}, \dots, X_{v_\ell}) | X_i}\|_{\infty, \mu^{\otimes (\ell+1)}} < \infty, \quad \forall \ell \leq j.$$

Let $\mathbb{X}_n = \{X_1, \dots, X_n\}$ and let $(\eta_n : n \in \mathbb{N})$ be a sequence in \mathbb{R}_+ with $\liminf_{n \rightarrow \infty} \eta_n > 0$.

Then there is a constant $C \in \mathbb{R}_+$ such that for each $A \in \mathfrak{S}$ and for each $r > 0$

$$\mathbb{E}[\mathcal{K}_j((\eta_n \mathbb{X}_n) \cap A, r)] \leq \mathbb{E}[\mathcal{K}_j(\eta_n \mathbb{X}_n, r; A)] \leq Cn^{j+1} \mu(\eta_n^{-1}A) \sup_{x \in S} \mu(B(x, 2\eta_n^{-1}r))^j \quad (4.2)$$

$$\text{and } \mathbb{E}[\mathcal{K}_j(\eta_n \mathbb{X}_n, r; \eta_n X_t)] \leq Cn^j \sup_{x \in S} \mu(B(x, 2\eta_n^{-1}r))^j. \quad (4.3)$$

In particular, if S is a subset of \mathbb{R}^p , $\eta_n = n^{1/p}$ and μ equals the Lebesgue measure, then (4.2) is of order $\mathcal{O}(|A|r^{pj})$ and (4.3) is of order $\mathcal{O}(r^{pj})$.

Proof. We only proof the statement in (4.2), the statement in (4.3) follows in the same fashion. The first inequality in

(4.2) is obvious. Thus, we only show the second one. Observe that

$$\mathcal{K}_j(\eta_n \mathbb{X}_n, r; A) \leq \sum_{i=1}^n \mathbb{1}\{\eta_n X_i \in A\} \sum_{\substack{(u_1, \dots, u_j) \in \{1, \dots, n\}^j \\ i \neq u_\ell \neq u_{\ell'}}} \prod_{\ell=1}^j \mathbb{1}\{d(X_i, X_{u_\ell}) \leq 2\eta_n^{-1} r\} \quad (4.4)$$

because the distance between any two points in a j -simplex in the Čech or the Vietoris-Rips complex $\mathcal{K}_j(\eta_n \mathbb{X}_n, r; A)$ is at most $2r$. On the one hand, for some $c_1 \in \mathbb{R}_+$

$$\mathbb{E} \left[\prod_{\ell=1}^j \mathbb{1}\{d(X_i, X_{u_\ell}) \leq 2\eta_n^{-1} r\} \middle| X_i \right] \leq c_1 \sup_{x \in S} \mu(B(x, 2\eta_n^{-1} r))^j, \quad (4.5)$$

and also $\#\{(u_1, \dots, u_j) \in \{1, \dots, n\}^j : i \neq u_\ell \neq u_{\ell'}, 1 \leq \ell, \ell' \leq j\} \leq n^j$. Moreover, on the other hand

$$\mathbb{E} [\mathbb{1}\{\eta_n X_i \in A\}] \leq c_2 \mu(\eta_n^{-1} A) \quad (4.6)$$

for some $c_2 \in \mathbb{R}_+$. Combining (4.5), (4.6) with (4.4) yields the conclusion. \square

The following result is well-known to topologists.

Lemma 4.3 (Geometric Lemma, Lemma 2.11 in Hiraoka et al. [2018]). *Let $\mathbb{X} \subseteq \mathbb{Y}$ be two finite point sets in S . Then $|\beta_q^{r,s}(\mathcal{K}(\mathbb{Y})) - \beta_q^{r,s}(\mathcal{K}(\mathbb{X}))| \leq \sum_{j=q}^{q+1} |\mathcal{K}_j(\mathbb{Y}, s) \setminus \mathcal{K}_j(\mathbb{X}, s)|$.*

We come to the proof of Theorem 2.6. Similar as in Yogeshwaran et al. [2017], we use a result of Chalker et al. [1999] to establish an exponential inequality without the need of bounding the martingale differences in the supremum-norm.

Proof of Theorem 2.6. Consider the natural filtration of the process X , $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$ for $i = 0, \dots, n$ with the convention that \mathcal{F}_0 is $\{\emptyset, S\}$. We rewrite the q th persistent Betti number in terms of martingale differences, viz.,

$$\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_n)) - \mathbb{E} [\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_n))] = \sum_{i=1}^n V_{n,i}, \quad (4.7)$$

where $V_{n,i} = \mathbb{E} [\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_n)) | \mathcal{F}_i] - \mathbb{E} [\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_n)) | \mathcal{F}_{i-1}]$. A result of Chalker et al. [1999] yields

$$\mathbb{P} \left(\left| \sum_{i=1}^n V_{n,i} \right| > b_1 \right) \leq 2 \exp \left(-\frac{b_1^2}{32nb_2^2} \right) + \left(1 + \frac{2 \sup_{1 \leq i \leq n} \|V_{n,i}\|_{\mathbb{P}, \infty}}{b_1} \right) \sum_{i=1}^n \mathbb{P} (\|V_{n,i}\| > b_2) \quad (4.8)$$

for any $b_1, b_2 \in \mathbb{R}_+$. Hence, it remains to compute bounds of $V_{n,i}$. In general, the difference of the q th Betti number is at most n^q for two point clouds $\mathbb{X}_{n,1}, \mathbb{X}_{n,2}$ of size n , i.e., $|\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_{n,1})) - \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{X}_{n,2}))| \leq n^q$. In particular $\|V_{n,i}\|_{\mathbb{P}, \infty}$ is bounded above by n^q times a constant.

Next, we investigate the probabilities on the right-hand side of (4.8). Define for $a \in S$ and $i = 1, \dots, n$

$$I_{n,i}(a) := \int_{S^{n-i}} \mathbb{P} (X_{i+1}^n \in dx_{i+1}^n | X_1, \dots, X_{i-1}, X_i = a) \beta_q^{r,s}(\mathcal{K}(\eta_n \{X_1^{i-1}, a, x_{i+1}^n\})).$$

Write ν_i for the conditional distribution of X_i given (X_1, \dots, X_{i-1}) on S , viz., $\nu_i(\cdot) = \mathbb{M}_{X_i | X_1^{i-1}}(X_1^{i-1}, \cdot)$.

Then, noting that the measure ν_i is random and using elementary calculations, we obtain for each $1 \leq i \leq n$

$$V_{n,i} = \int_{S^{n-i}} \mathbb{P} (X_{i+1}^n \in dx_{i+1}^n | X_1, \dots, X_{i-1}, X_i) \beta_q^{r,s}(\mathcal{K}(\eta_n \{X_1^{i-1}, X_i, x_{i+1}^n\}))$$

$$\begin{aligned}
& - \int_S \mathbb{P}(X_i \in dx_i | X_1, \dots, X_{i-1}) \int_{S^{n-i}} \mathbb{P}(X_{i+1}^n \in dx_{i+1}^n | X_1, \dots, X_{i-1}, X_i = x_i) \\
& \quad \beta_q^{r,s}(\mathcal{K}(\eta_n \{X_1^{i-1}, x_i, x_{i+1}^n\})) \\
& \leq \operatorname{ess\,sup}_{\text{w.r.t. } \nu_i} I_{n,i}(\cdot) - \operatorname{ess\,inf}_{\text{w.r.t. } \nu_i} I_{n,i}(\cdot) \quad a.s.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary but fixed. Choose $a^*, b^* \in S$ such that $I_{n,i}(a^*) \geq \operatorname{ess\,sup}_{\text{w.r.t. } \nu_i} I_{n,i}(\cdot) - \varepsilon/2$ and $I_{n,i}(b^*) \leq \operatorname{ess\,inf}_{\text{w.r.t. } \nu_i} I_{n,i}(\cdot) + \varepsilon/2$, note that both quantities on the right-hand sides are finite. Consider the Marton coupling of (X_1, \dots, X_n) and write $\mathbb{X}_n^{(X_1, \dots, X_{i-1}, a^*, b^*)}$ for the point cloud associated to the coupling element $X^{(X_1, \dots, X_{i-1}, a^*, b^*)}$. The same notation can be used for the point cloud obtained from the counterpart $X'^{(X_1, \dots, X_{i-1}, a^*, b^*)}$. Consequently,

$$\begin{aligned}
V_{n,i} - \varepsilon & \leq I_{n,i}(a^*) - I_{n,i}(b^*) \\
& = \mathbb{E} \left[\beta_q^{r,s} \left(\mathcal{K}(\eta_n \mathbb{X}_n^{(X_1, \dots, X_{i-1}, a^*, b^*)}) \right) - \beta_q^{r,s} \left(\mathcal{K}(\eta_n \mathbb{X}'^{(X_1, \dots, X_{i-1}, a^*, b^*)}) \right) \mid X_1, \dots, X_{i-1} \right] \\
& = \int_{S^{n-i} \times S^{n-i}} \mathbb{P}_{(X^{(X_1, \dots, X_{i-1}, a^*, b^*)}, X'^{(X_1, \dots, X_{i-1}, a^*, b^*)})} (d(y_{i+1}^n, y'_{i+1}{}^n)) \\
& \quad \times \left\{ \beta_q^{r,s} \left(\mathcal{K}(\eta_n \{X_1^{i-1}, a^*, y_{i+1}^n\}) \right) - \beta_q^{r,s} \left(\mathcal{K}(\eta_n \{X_1^{i-1}, b^*, y'_{i+1}{}^n\}) \right) \right\}, \tag{4.9}
\end{aligned}$$

(by abusing the notation slightly). We write $\mathbb{Y}_{n,i} = \{X_1^{i-1}, a^*, y_{i+1}^n\}$ and $\mathbb{Y}'_{n,i} = \{X_1^{i-1}, b^*, y'_{i+1}{}^n\}$. Moreover, the point clouds $\mathbb{Y}_{n,i}$ and $\mathbb{Y}'_{n,i}$ in (4.9) differ at most in $n - i + 1$ entries for each i . These entries are $\{a^*, y_{i+1}, \dots, y_n\}$ and $\{b^*, y'_{i+1}, \dots, y'_n\}$. Thus, we can transform $\mathbb{Y}_{n,i}$ into $\mathbb{Y}'_{n,i}$ in $n - i + 1$ steps exchanging one entry in each step, i.e., we consider the transformations

$$\mathbb{Y}_{n,i}^{(1)} = \mathbb{Y}_{n,i} \rightsquigarrow \mathbb{Y}_{n,i}^{(2)} \rightsquigarrow \dots \rightsquigarrow \mathbb{Y}_{n,i}^{(n-i+2)} = \mathbb{Y}'_{n,i}, \tag{4.10}$$

where $\mathbb{Y}_{n,i}^{(\ell)} = \{X_1^{i-1}, b^*, y_{i+1}^{i+\ell-2}, y_{i+\ell-1}^n\}$, for $\ell = 2, \dots, n - i + 2$.

Using this definition, the difference of Betti numbers in (4.9) is bounded above by

$$\left| \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i})) - \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}'_{n,i})) \right| \leq \sum_{\ell=1}^{n-i+1} \left| \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i}^{(\ell+1)})) - \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i}^{(\ell)})) \right|. \tag{4.11}$$

The symmetric difference $\mathbb{Y}_{n,i}^{(\ell+1)} \Delta \mathbb{Y}_{n,i}^{(\ell)}$ is at most $\{a^*, b^*\}$ for $\ell = 1$ and $\{y_{\ell+i-1}, y'_{\ell+i-1}\}$ for $\ell = 2, \dots, n - i + 1$. Let $\ell \in \{2, \dots, n - i + 1\}$. Clearly, if $y_{\ell+i-1} = y'_{\ell+i-1}$, then $|\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i}^{(\ell+1)})) - \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i}^{(\ell)}))| = 0$. If $y_{\ell+i-1} \neq y'_{\ell+i-1}$, we can use Lemma 4.3 to obtain

$$\begin{aligned}
\left| \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i}^{(\ell+1)})) - \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i}^{(\ell)})) \right| & \leq \sum_{j=q, q+1} \mathcal{K}_j \left(\eta_n (\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}), s \right) \setminus \mathcal{K}_j \left(\eta_n \mathbb{Y}_{n,i}^{(\ell+1)}, s \right) \\
& \quad + \sum_{j=q, q+1} \mathcal{K}_j \left(\eta_n (\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}), s \right) \setminus \mathcal{K}_j \left(\eta_n \mathbb{Y}_{n,i}^{(\ell)}, s \right) \\
& \leq \sum_{j=q, q+1} \mathcal{K}_j \left(\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}, \eta_n^{-1} s; \{y_{\ell+i-1}\} \right) \\
& \quad + \sum_{j=q, q+1} \mathcal{K}_j \left(\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}, \eta_n^{-1} s; \{y'_{\ell+i-1}\} \right), \tag{4.12}
\end{aligned}$$

where we use for the derivation of the last inequality also the scaling relation $\mathcal{K}_j(\eta \mathbb{X}, r) = \mathcal{K}_j(\mathbb{X}, \eta^{-1} r)$, which is valid for the Čech and the Vietoris-Rips complex.

A similar argument applies to $|\beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i}^{(2)})) - \beta_q^{r,s}(\mathcal{K}(\eta_n \mathbb{Y}_{n,i}^{(1)}))|$, which admits the same bound as in (4.12) using the points a^*, b^* instead of $y_{\ell+i-1}$ and $y'_{\ell+i-1}$.

Write $N = N(r, S, d)$ for the r -covering number of S w.r.t. d from Condition 2.3. Use the family of coverings $\{\{B(w_j, r) : 1 \leq j \leq N\} : r > 0\}$ to define for each $r > 0$ and each $u > 0$ the set

$$\mathbb{A}_{n,u}(r) := \{x \in S^n : \#[B(w_j, r) \cap x] \leq u, \forall 1 \leq j \leq N(r, S, d)\}. \quad (4.13)$$

Intuitively, $\mathbb{Y}_{n,i} \in \mathbb{A}_{n,u}(\eta_n^{-1}s)$ means that this point cloud is regularly distributed on S if $u \ll n$. Consequently, (4.9) is at most

$$\begin{aligned} & \int_{S^{n-i} \times S^{n-i}} \mathbb{P}_{(X^{(X_1, \dots, X_{i-1}, a^*, b^*)}, X'^{(X_1, \dots, X_{i-1}, a^*, b^*)})} (d(y_{i+1}^n, y'_{i+1}^n)) \left\{ \right. \\ & \quad \left[\mathbb{1}\{\mathbb{Y}_{n,i} \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s)\} + \mathbb{1}\{\mathbb{Y}'_{n,i} \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s)\} \right] \times n^q \\ & \quad + \mathbb{1}\{\mathbb{Y}_{n,i} \in \mathbb{A}_{n,u}(4\eta_n^{-1}s), \mathbb{Y}'_{n,i} \in \mathbb{A}_{n,u}(4\eta_n^{-1}s)\} \times \left[\sum_{\ell=2}^{n-i+1} \left(\mathbb{1}\{y_{\ell+i-1} \neq y'_{\ell+i-1}\} \right. \right. \\ & \quad \left. \left. \sum_{j=q, q+1} \left[\mathcal{K}_j(\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}, \eta_n^{-1}s; \{y_{\ell+i-1}\}) + \mathcal{K}_j(\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}, \eta_n^{-1}s; \{y'_{\ell+i-1}\}) \right] \right. \right. \\ & \quad \left. \left. + \sum_{j=q, q+1} \left[\mathcal{K}_j(\mathbb{Y}_{n,i}^{(2)} \cup \mathbb{Y}_{n,i}^{(1)}, \eta_n^{-1}s; \{a^*\}) + \mathcal{K}_j(\mathbb{Y}_{n,i}^{(2)} \cup \mathbb{Y}_{n,i}^{(1)}, \eta_n^{-1}s; \{b^*\}) \right] \right] \right\}. \end{aligned} \quad (4.14)$$

Next, we need to bound above the number of j -simplices provided that the point clouds $\mathbb{Y}_{n,i}$ and $\mathbb{Y}'_{n,i}$ are regularly distributed on S in that $\mathbb{Y}_{n,i} \in \mathbb{A}_{n,u}(4\eta_n^{-1}s)$ and $\mathbb{Y}'_{n,i} \in \mathbb{A}_{n,u}(4\eta_n^{-1}s)$.

It suffices to consider the term $\mathcal{K}_j(\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}, \eta_n^{-1}s; \{y_{\ell+i-1}\})$. Clearly, if both $\mathbb{Y}_{n,i}$ and $\mathbb{Y}'_{n,i}$ are in $\mathbb{A}_{n,u}(4\eta_n^{-1}s)$, then each point cloud of the type $\mathbb{Y}_{n,i}^{(\ell)}$ is in $\mathbb{A}_{n,2u}(4\eta_n^{-1}s)$ and $\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}$ is in $\mathbb{A}_{n,2u+1}(4\eta_n^{-1}s)$.

Observe that a j -simplex in the filtration $\mathcal{K}(\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}, \eta_n^{-1}s)$ has a diameter of at most $2\eta_n^{-1}s$. Thus, a j -simplex with a node in a point $y_{\ell+i-1}$ lies in the $(2\eta_n^{-1}s)$ -neighborhood of $y_{\ell+i-1}$. By Condition 2.3 there is a covering $\{B(w_k, 4\eta_n^{-1}s) : 1 \leq j \leq N(4\eta_n^{-1}s, S, d)\}$ of S . Consequently, there is a $1 \leq k \leq N(4\eta_n^{-1}s, S, d)$ such that the neighborhood $B(y_{\ell+i-1}, 2\eta_n^{-1}s)$ is contained in $B(w_k, 4\eta_n^{-1}s)$.

Consequently, in the case where $\mathbb{Y}_{n,i}$ and $\mathbb{Y}'_{n,i}$ are both in $\mathbb{A}_{n,u}(4\eta_n^{-1}s)$, each term of the type $\mathcal{K}_j(\mathbb{Y}_{n,i}^{(\ell+1)} \cup \mathbb{Y}_{n,i}^{(\ell)}, \eta_n^{-1}s; \{y_{\ell+i-1}\})$ is bounded above by a constant (which depends on s and q) times u^{q+1} . Thus, we obtain an upper bound of the following type for the integral in (4.14)

$$\begin{aligned} & \int_{S^{n-i} \times S^{n-i}} \mathbb{P}_{([X^{(X_1, \dots, X_{i-1}, a^*, b^*)}]_{i+1}^n, [X'^{(X_1, \dots, X_{i-1}, a^*, b^*)}]_{i+1}^n)} (d(y_{i+1}^n, y'_{i+1}^n)) \\ & \quad \times \left\{ n^q (\mathbb{1}\{\mathbb{Y}_{n,i} \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s)\} + \mathbb{1}\{\mathbb{Y}'_{n,i} \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s)\}) + c_1 u^{q+1} \sum_{j=i}^n \mathbb{1}\{y_j \neq y'_j\} \right\} \\ & = n^q (\mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s) | X_1^{i-1}, X_i = a^*) + \mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s) | X_1^{i-1}, X_i = b^*)) \\ & \quad + c_1 u^{q+1} \sum_{j=i}^n \mathbb{P}(X_j^{(X_1, \dots, X_{i-1}, a^*, b^*)} \neq X'_j{}^{(X_1, \dots, X_{i-1}, a^*, b^*)}). \end{aligned} \quad (4.15)$$

Note that c_1 depends on s and q but neither on i or n . Furthermore, for the choice $u = n^\gamma$ the last term in (4.15) is in $\mathcal{O}(n^{\gamma(q+1)})$ a.s. uniformly in i and a^*, b^* due to the condition from 2.3 on the mixing matrix, which implies that

$$\sup_{n \in \mathbb{N}_+} \sup_{i \in \{1, \dots, n\}} \sum_{j=1}^n \mathbb{P}(X_j^{(X_1, \dots, X_{i-1}, a^*, b^*)} \neq X'_j{}^{(X_1, \dots, X_{i-1}, a^*, b^*)}) \leq \sup_{n \in \mathbb{N}_+} \sup_{i \in \{1, \dots, n\}} \sum_{j=1}^n \Gamma_{i,j}^{(n)} < \infty \quad a.s.$$

Moreover as the choice of ε was arbitrary, this last bound from (4.15) is also true for the limit $\text{ess sup}_{\text{w.r.t. } \nu_i} I_{n,i}(\cdot) -$

ess inf $\text{w.r.t. } \nu_i I_{n,i}(\cdot)$ uniformly in i .

We return to (4.9). If we choose $b_2 = c_2 n^{\gamma(q+1)}$ for a certain constant $c_2 = c_2(s, q)$, the last term in (4.15) and the approximation by ε are negligible. Thus, for certain constants c_3, \dots, c_5 depending on s and q

$$\begin{aligned} \mathbb{P}(|V_{n,i}| > b_2) &\leq c_3 \left\{ \mathbb{P} \left[n^q \mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s) \mid X_1^{i-1}, X_i = a^*) > c_4 b_2 \right] \right. \\ &\quad \left. + \mathbb{P} \left[n^q \mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s) \mid X_1^{i-1}, X_i = b^*) > c_4 b_2 \right] \right\} \\ &\leq c_5 \frac{n^q}{n^{\gamma(q+1)}} \left[\mathbb{E} \left[\mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s) \mid X_1^{i-1}, X_i = a^*) \right] \right. \\ &\quad \left. + \mathbb{E} \left[\mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s) \mid X_1^{i-1}, X_i = b^*) \right] \right], \end{aligned} \quad (4.16)$$

where we use Markov's inequality in the last step. We bound above both expectations in (4.16) using the inequality

$$\mathbb{E} \left[\mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u}(4\eta_n^{-1}s) \mid X_1^{i-1}, X_i = a) \right] \leq \mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u-1}(4\eta_n^{-1}s)),$$

which holds for any state $a \in S$. Consider the covering $\{B(w_j, 4\eta_n^{-1}s) : 1 \leq j \leq N\}$ of S . The covering condition from (2.6), implies that $\log N = \mathcal{O}((\log n)^{\bar{\alpha}})$ for some $\bar{\alpha} \in \mathbb{R}_+$. Moreover, using (4.1) from Lemma 4.1, we find that

$$\begin{aligned} \mathbb{P}(\mathbb{X}_n \notin \mathbb{A}_{n,u-1}(4\eta_n^{-1}s)) &= \mathbb{P}(X_1^n \cap B(w_j, 4\eta_n^{-1}s) > n^\gamma - 1 \text{ for one } j = 1, \dots, N) \\ &\leq \exp(c_6(\log n)^{\bar{\alpha}}) \times \max_{j \in \mathbb{N}} \mathbb{P}(X_1^n \cap B(w_j, 4\eta_n^{-1}s) > n^\gamma - 1) \\ &\leq \exp(c_6(\log n)^{\bar{\alpha}} - n^\gamma + 1 + (e-1)f^* n \sup_{w \in S} \mu(B(w, 4\eta_n^{-1}s))) \\ &\leq c_7 \exp(-n^\gamma + c_8(\log n)^{\alpha \vee \bar{\alpha}}), \end{aligned}$$

where we use (2.5) for the derivation of the last inequality and where $c_6, \dots, c_8 \in \mathbb{R}_+$ depend on s and q . Combining this last inequality with (4.16), we see that

$$\mathbb{P}(|V_{n,i}| > b_2) = \mathcal{O}\left(n^{q(1-\gamma)-\gamma} \exp(-n^\gamma + c_7(\log n)^{\alpha \vee \bar{\alpha}})\right).$$

Moreover, inserting this result in (4.8) for the above choice of b_2 and $b_1 = n^a t$, yields

$$\begin{aligned} \mathbb{P}\left(\left|\sum_{i=1}^n V_{n,i}\right| > n^a t\right) &\leq 2 \exp(-C_1 n^{2a-1-2\gamma q} t^2) \\ &\quad + C_2 (1 + n^{q-a} t^{-1}) n^{(q+1)(1-\gamma)} \exp(-n^\gamma + C_3(\log n)^{\alpha \vee \bar{\alpha}}) \end{aligned}$$

for three constants $C_1, \dots, C_3 \in \mathbb{R}_+$ depending on s and q but neither on n nor on t . \square

Proof of Theorem 2.7. In the first step of the proof, we show that

$$n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/d} \mathbb{X}_n)) \right] = \sum_{i=1}^{m^p} n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/d}(\mathbb{X}_n \cap A_i))) \right] + o(1), \quad n \rightarrow \infty.$$

Define a filtration, which is the union of the single filtrations when restricted to the cubes A_i , by

$$\mathring{\mathcal{K}}(n^{1/p}\mathbb{X}_n, r) = \bigcup_{i=1}^{m^p} \mathcal{K}((n^{1/p}(\mathbb{X}_n \cap A_i), r), \quad r \geq 0.$$

Since this union is of disjoint complexes, $\beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/p}\mathbb{X}_n)) = \sum_{i=1}^{m^p} \beta_q^{r,s}(\mathcal{K}((n^{1/p}(\mathbb{X}_n \cap A_i)))$. Write $A^{(\varepsilon)}$ for the ε -offset of $A \in \mathcal{B}(\mathbb{R}^p)$, which is the set of points x with a distance of at most ε to A . We use Lemma 4.3 and Lemma 4.2 to arrive at

$$\begin{aligned} & n^{-1} \mathbb{E} \left[\left| \beta_q^{r,s}(\mathcal{K}((n^{1/p}\mathbb{X}_n)) - \beta_q^{r,s}(\mathring{\mathcal{K}}(n^{1/p}\mathbb{X}_n)) \right| \right] \\ & \leq n^{-1} \sum_{j=k, k+1} \mathbb{E} \left[\mathcal{K}_j \left((n^{1/p}\mathbb{X}_n) \cap \left(\bigcup_{i=1}^{m^p} (\partial(n^{1/p}A_i))^{(2s)} \right), s \right) \right] \leq c_1 m^p n^{(p-1)/p-1} = o(1), \quad n \rightarrow \infty, \end{aligned}$$

where c_1 depends on p, s . So, we can consider the expectation on the blocks A_i instead.

From now let $i \in \{1, \dots, m^p\}$ be an arbitrary but fixed index. Write $\ell_{i,1}, \dots, \ell_{i,p}$ for the edge lengths of A_i . So that $|A_i|$ equals $\prod_{j=1}^p \ell_{i,j}$. Also write M_i for the diagonal matrix $\text{diag}(\ell_{i,j} : j = 1, \dots, p)$. Note that $\det(M_i) = |A_i|$.

In the second step, we use McDiarmid's inequality from Theorem A.2. Set $S_{n,i} = \sum_{t=1}^n \mathbb{1}\{X_t \in A_i\}$ and $h(n) = n^{3/4}$. Since (X_1, \dots, X_n) admits a Marton coupling which satisfies (2.3), we can apply Theorem A.2 to arrive at

$$\mathbb{P}(|S_{n,i} - \mathbb{E}[S_{n,i}]| > h(n)) \leq 2 \exp(-c_2 h(n)^2 n^{-1}),$$

for a certain constant $c_2 \in \mathbb{R}_+$, which does not depend on n .

Using the definition $I_{n,i} = [-h(n) + \mathbb{E}[S_{n,i}], h(n) + \mathbb{E}[S_{n,i}]]$ and the fact that the Betti numbers are polynomially bounded, we obtain

$$n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p}(\mathbb{X}_n \cap A_i))) \right] = \sum_{k \in I_{n,i}} n^{-1} \mathbb{E} \left[\mathbb{1}\{S_{n,i} = k\} \beta_q^{r,s}(\mathcal{K}(n^{1/p}(\mathbb{X}_n \cap A_i))) \right] + o(1), \quad n \rightarrow \infty.$$

Write $\mu_{n,i} = \lfloor \mathbb{E}[S_{n,i}] \rfloor = \lfloor n\alpha_i |A_i| \rfloor$, then it follows from Lemma 4.9 in Krebs and Polonik [2019] that for each $0 \leq r \leq s$

$$\limsup_{n \rightarrow \infty} \sup_{k \in I_{n,i}} n^{-1} \left| \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p}(X'_1, \dots, X'_k))) \right] - \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p}(X'_1, \dots, X'_{\mu_{n,i}}))) \right] \right| = 0, \quad (4.17)$$

where the X'_t are independent and uniformly distributed on $[0, 1]^p$. We will use this relation later.

In the third step, we study the success runs of $(X_t : 1 \leq t \leq n)$ and the sum $S_{n,i}$. If an X_t falls in A_i , we term this a success and a failure otherwise. Consider a path with exactly k successes $J = (\mathcal{F}_1, \mathcal{S}_1, \mathcal{F}_2, \dots, \mathcal{F}_v, \mathcal{S}_v, \mathcal{F}_{v+1}) \in \{0, 1\}^n$, where $v \leq k$ and where each \mathcal{S}_i is a sequence of 1's and each \mathcal{F}_i a sequence of 0's (potentially \mathcal{F}_1 and \mathcal{F}_{v+1} 0 length) for $i = 1, \dots, v$, (resp. $v+1$).

Consider the expectation on this path J . For this write J^* for the index set which contains the positions in J that mark a success. Write \mathbb{M}_i for the conditional distribution of X_i given the past X_1, \dots, X_{i-1} . Then

$$\begin{aligned} & n^{-1} \mathbb{E} \left[\mathbb{1}\{X_1^n \in J\} \beta_q^{r,s}(\mathcal{K}(n^{1/p}(\mathbb{X}_n \cap A_i))) \right] \\ & = n^{-1} \int_{[0,1]^p} \mathbb{P}_{X_1}(dx_1) \int_{[0,1]^p} \mathbb{M}_2(dx_2|x_1) \dots \int_{[0,1]^p} \mathbb{M}_n(dx_n|x_1, \dots, x_{n-1}) \\ & \quad \times \beta_q^{r,s}(\mathcal{K}(n^{1/p}(x_1^n \cap A_i))) \prod_{i \in J^*} \mathbb{1}\{x_\ell \in A_i\} \prod_{i \in J \setminus J^*} \mathbb{1}\{x_\ell \notin A_i\}. \end{aligned} \quad (4.18)$$

Consider the situation for the last success which is given at a position t^* . Note that each \mathbb{M}_t admits a conditional density f_t because the distribution of X_t on each block A_j , $j = 1, \dots, m^p$, is uniform and independent of the past observations X_1, \dots, X_{t-1} given that X_t falls in the block A_j . So this $f_t(x_t|x_1, \dots, x_{t-1})$ is constant for all x_t from a block A_j . Due to the blocked structure of the conditional densities of X and the invariance property from (2.10), the contribution of the observations X_{t^*}, \dots, X_n to the integral in (4.18) is then

$$\begin{aligned} & \int_{[0,1]^p} \mathbb{M}_{t^*}(dx_{t^*}|x_1, \dots, x_{t^*-1}) \beta_q^{r,s}(\mathcal{K}(n^{1/p}(x_{J^* \setminus \{t^*\}}, x_{t^*}))) \mathbb{P}(X_{t^*+1} \notin A_i, \dots, X_n \notin A_i | x_1, \dots, x_{t^*}) \\ &= f_{t^*}(z_{t^*}|x_1, \dots, x_{t^*-1}) \mathbb{P}(X_{t^*+1} \notin A_i, \dots, X_n \notin A_i | x_1, \dots, x_{t^*-1}, z_{t^*}) \int_{[0,1]^p} dx_{t^*} \beta_q^{r,s}(\mathcal{K}(n^{1/p}(x_{J^* \setminus \{t^*\}}, x_{t^*}))) \\ &= \mathbb{P}(X_{t^*} \in A_i, X_{t^*+1} \notin A_i, \dots, X_n \notin A_i | x_1, \dots, x_{t^*-1}) \int_{[0,1]^p} dx_{t^*} |A_i|^{-1} \beta_q^{r,s}(\mathcal{K}(n^{1/p}(x_{J^* \setminus \{t^*\}}, x_{t^*}))), \end{aligned}$$

where z_{t^*} is an arbitrary but fixed element of A_i . Using this conditional independence argument, one obtains for (4.18)

$$\begin{aligned} & n^{-1} \mathbb{E} \left[\mathbb{1}\{X_1^n \in J\} \beta_q^{r,s}(\mathcal{K}(n^{1/p}(\mathbb{X}_n \cap A_i))) \right] \\ &= n^{-1} \mathbb{P}(X_1^n \in J) \int_{A_i} \dots \int_{A_i} dx_1 \dots dx_k |A_i|^{-k} \beta_q^{r,s}(n^{1/p}(x_1, \dots, x_k)) \\ &= \mathbb{P}(X_1^n \in J) n^{-1} \mathbb{E} \left[\beta_q^{r,s}(n^{1/p}\{M_i X'_1, \dots, M_i X'_k\}) \right], \end{aligned} \quad (4.19)$$

where the equality follows from the dilatation rules of the expectation of persistent Betti numbers computed from the Čech or Vietoris-Rips filtration and where the X'_t are independent and uniformly distributed on $[0, 1]^p$.

Moreover, using the uniform approximation result from (4.17) shows that (4.19) equals

$$\mathbb{P}(X_1^n \in J) n^{-1} \left(\mathbb{E} \left[\beta_q^{r,s}(n^{1/p}\{M_i X'_1, \dots, M_i X'_{\mu_{n,i}}\}) \right] + o(1) \right), \quad n \rightarrow \infty.$$

Note that the remainder is uniform in $k \in I_{n,i}$. Furthermore, using again the dilatation rules, we obtain for the main term of this last line

$$\begin{aligned} & \mathbb{P}(X_1^n \in J) (\alpha_i |A_i|) \frac{\alpha_i^{-1} |A_i|^{-1}}{n} \mathbb{E} \left[\beta_q^{\alpha_i^{1/p}(r,s)}((n\alpha_i |A_i|)^{1/p} \{(M_i |A_i|^{-1/p}) X'_1, \dots, (M_i |A_i|^{-1/p}) X'_{\mu_{n,i}}\}) \right] \\ &= \mathbb{P}(X_1^n \in J) \left((\alpha_i |A_i|) \hat{b}_q(\alpha_i^{1/p}(r,s)) + o(1) \right), \quad n \rightarrow \infty, \end{aligned}$$

where the last equality follows as in the proof of Lemma 10 in Divol and Polonik [2018]. Summing over all paths J with exactly k successes, over all $k \in I_{n,i}$ and over all $i = 1, \dots, m^p$ yields then the conclusion, viz.,

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \mathbb{X}_n)) \right] = \sum_{i=1}^{m^p} (\alpha_i |A_i|) \hat{b}_q(\alpha_i^{1/p}(r,s)) = \mathbb{E} \left[\hat{b}_q(\kappa(X_t)^{1/p}(r,s)) \right].$$

This proves (2.11). Combining this last statement with Theorem 2.6 and the Borel-Cantelli-Lemma shows (2.12). Consequently, the proof is complete. \square

Proof of Theorem 2.8. In the proof, we sometimes abuse the notation slightly in order to keep formulas shorter. This is, we also write $\mathcal{K}(U_1^n, r)$ for the simplicial complex $\mathcal{K}(\{U_1, \dots, U_n\}, r)$ of a vector U_1, \dots, U_n at filtration time r to save space. Other related expressions are also abbreviated in this way.

In the first step of the proof, we construct a discrete Markov chain of order m , \tilde{X} , which approximates X closely. To this end, let $\varepsilon > 0$ be arbitrary but fixed. We use a discrete density function g_ε , which is an approximation of the joint density g of (X_1, \dots, X_m) . We write f_{t-1} for the conditional density of X_t given X_{t-1}, \dots, X_1 . Since we assume that

the process X is a Markov chain of order m , we are actually dealing with the conditional densities $f_0 \equiv \kappa, f_1, \dots, f_m$ only. Using the approximation g_ε , we also obtain approximations $f_{\varepsilon,1} =: \kappa_\varepsilon, f_{\varepsilon,2}, \dots, f_{\varepsilon,m+1}$. We choose the precision between g and g_ε sufficiently high (in the $\|\cdot\|_\infty$ -norm) such that

$$|f_{\varepsilon,t}(x_i|x_{i-t}^{i-1}) - f_t(x_i|x_{i-t}^{i-1})| \leq \varepsilon, \quad \forall x = (x_{i-t}, \dots, x_{i-1}, x_i) \in [0, 1]^{(t+1)p}, \quad \forall t = 0, \dots, m. \quad (4.20)$$

Thus, at each step of the evolution of the Markov chain, we can approximate each conditional density with a discrete conditional density with a precision of at least ε (measured in the total variation distance). Note that this is possible because we assume that $\inf\{g(z) : z \in [0, 1]^{(m+1)p}\} > 0$, so that all conditional densities are well defined.

We write \tilde{X} for the Markov chain of order m obtained from the above ε -approximation scheme, note that we can also choose g_ε to be strictly positive. In particular, this implies that \tilde{X} satisfies the assumptions of Theorem 2.8 because it is uniformly geometrically ergodic, see Meyn and Tweedie [2012] Theorem 16.0.2. Clearly, also X is uniformly geometrically ergodic, hence, X admits a Marton coupling which satisfies (2.3), see also Example 2.2 and Paulin [2015] Proposition 2.4. This means the mixing matrix $\Gamma^{(n)}$ of X_1, \dots, X_n satisfies

$$\sup_{n \in \mathbb{N}_+} \sum_{j=1}^n \Gamma_{1,j}^{(n)} < \infty. \quad (4.21)$$

In the second step, we use the decomposition

$$\begin{aligned} & \left| n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \mathbb{X}_n)) \right] - \mathbb{E} \left[\hat{b}_q(\kappa(Y)^{1/p}(r, s)) \right] \right| \\ & \leq n^{-1} \left| \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \mathbb{X}_n)) \right] - \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \tilde{\mathbb{X}}_n)) \right] \right| \end{aligned} \quad (4.22)$$

$$+ \left| n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \tilde{\mathbb{X}}_n)) \right] - \mathbb{E} \left[\hat{b}_q(\kappa_\varepsilon(Y_\varepsilon)^{1/p}(r, s)) \right] \right| \quad (4.23)$$

$$+ \left| \mathbb{E} \left[\hat{b}_q(\kappa_\varepsilon(Y_\varepsilon)^{1/p}(r, s)) \right] - \mathbb{E} \left[\hat{b}_q(\kappa(Y)^{1/p}(r, s)) \right] \right|, \quad (4.24)$$

where the random variables Y_ε (resp. Y) have density κ_ε (resp. κ).

If ε converges to 0, also (4.24) converges to 0, see also Divol and Polonik [2018]. Moreover, from Theorem 2.7, we conclude that (4.23) converges to 0 as n tends to ∞ .

Consequently, we remains to show that (4.22) is at most $C\varepsilon$ uniformly in n , where the constant C does neither depend on the choice of the approximation parameter ε nor on n . For this we rewrite (4.22) as

$$\begin{aligned} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \mathbb{X}_n)) \right] &= \int dx_1 f_0(x_1) \int dx_2 f_1(x_2 | x_1) \dots \int dx_{m+1} f_m(x_{m+1} | x_1^m) \\ & \int dx_{m+2} f_m(x_{m+2} | x_2^{m+1}) \dots \int dx_n f_m(x_n | x_{n-m}^{n-1}) \beta_q^{r,s}(\mathcal{K}(n^{1/p} x_1^n)), \end{aligned} \quad (4.25)$$

$$\begin{aligned} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \tilde{\mathbb{X}}_n)) \right] &= \int dx_1 f_{\varepsilon,0}(x_1) \int dx_2 f_{\varepsilon,1}(x_2 | x_1) \dots \int dx_{m+1} f_{\varepsilon,m}(x_{m+1} | x_1^m) \\ & \int dx_{m+2} f_{\varepsilon,m}(x_{m+2} | x_2^{m+1}) \dots \int dx_n f_{\varepsilon,m}(x_n | x_{n-m}^{n-1}) \beta_q^{r,s}(\mathcal{K}(n^{1/p} x_1^n)). \end{aligned} \quad (4.26)$$

We transform (4.25) in (4.26) in n -steps using in each step a specific coupling. This shows then that at each step, the difference is of order ε uniformly in n . Consequently, also the difference between (4.25) and (4.26), when divided by n , is of order ε .

First, we write the difference between (4.25) and (4.26) as a telescopic sum (the exchanged factor is given in square

parentheses)

$$\begin{aligned}
& \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p}\mathbb{X}_n)) - \beta_q^{r,s}(\mathcal{K}(n^{1/p}\tilde{\mathbb{X}}_n)) \right] \\
&= \int dx_1 [f_0(x_1) - f_{\varepsilon,0}(x_1)] \int dx_2 f_1(x_2 | x_1) \dots \int dx_{m+1} f_m(x_{m+1} | x_1^m) \\
&\quad \int dx_{m+2} f_m(x_{m+2} | x_2^{m+1}) \dots \int dx_n f_m(x_n | x_{n-m}^{n-1}) \beta_q^{r,s}(\mathcal{K}(n^{1/p}x_1^n)) \\
&+ \int dx_1 f_{\varepsilon,0}(x_1) \int dx_2 [f_1(x_2 | x_1) - f_{\varepsilon,1}(x_2 | x_1)] \dots \int dx_{m+1} f_m(x_{m+1} | x_1^m) \\
&\quad \int dx_{m+2} f_m(x_{m+2} | x_2^{m+1}) \dots \int dx_n f_m(x_n | x_{n-m}^{n-1}) \beta_q^{r,s}(\mathcal{K}(n^{1/p}x_1^n)) \\
&+ \dots \\
&+ \int dx_1 f_{\varepsilon,0}(x_1) \int dx_2 f_{\varepsilon,1}(x_2 | x_1) \dots \int dx_{m+1} f_{\varepsilon,m}(x_{m+1} | x_1^m) \\
&\quad \int dx_{m+2} f_{\varepsilon,m}(x_{m+2} | x_2^{m+1}) \dots \int dx_t [f_m(x_t | x_{t-m}^{t-1}) - f_{\varepsilon,m}(x_t | x_{t-m}^{t-1})] \dots \\
&\quad \dots \int dx_n f_m(x_n | x_{n-m}^{n-1}) \beta_q^{r,s}(\mathcal{K}(n^{1/p}x_1^n)) \\
&+ \dots \\
&+ \int dx_1 f_{\varepsilon,0}(x_1) \int dx_2 f_{\varepsilon,1}(x_2 | x_1) \dots \int dx_{m+1} f_{\varepsilon,m}(x_{m+1} | x_1^m) \\
&\quad \int dx_{m+2} f_{\varepsilon,m}(x_{m+2} | x_2^{m+1}) \dots \int dx_n [f_m(x_n | x_{n-m}^{n-1}) - f_{\varepsilon,m}(x_n | x_{n-m}^{n-1})] \beta_q^{r,s}(\mathcal{K}(n^{1/p}x_1^n)). \tag{4.27}
\end{aligned}$$

Each integral in the sum can be interpreted as a difference between the expectation of two persistent Betti numbers of two coupled processes. We explain this coupling in three steps, see also (4.27) which shows the general situation. First, the t th coupling starts with two joint processes $\tilde{Z}_{t,1} = Z'_{t,1}, \dots, \tilde{Z}_{t,t-1} = Z'_{t,t-1}$, which has the same distribution as the stationary discrete Markov chain \tilde{X} from time 1 to $t-1$.

Second, at time t , we simulate a random variable $Z'_{t,t}$ using the conditional density f_i (where the index i depends on the position of t). Also at time t , we simulate a random variable $\tilde{Z}_{t,t}$ using the conditional density $f_{\varepsilon,i}$. Note that $Z'_{t,t}$ and $\tilde{Z}_{t,t}$ can be coupled such that $\mathbb{P}(Z'_{t,t} \neq \tilde{Z}_{t,t} | \tilde{Z}_{t,1}, \dots, \tilde{Z}_{t,t-1}) \leq 2\varepsilon$ a.s. because of the choices in (4.20).

Third, we find two chains $Z'_{t,j}$ and $\tilde{Z}_{t,j}$, $j = t+1, \dots, n$, using the conditional densities f_i such that the single elements at time j satisfy $\mathbb{P}(Z'_{t,j} \neq \tilde{Z}_{t,j} | \tilde{Z}_{t,1}, \dots, \tilde{Z}_{t,t-1}, \tilde{Z}_{t,t}, Z'_{t,t}) \leq \Gamma_{t,j}^{(n)}$. This last inequality follows from the properties of the Marton coupling, see (4.21).

In the following, we show that there is a constant C such that for each $t \in \{1, \dots, n - \lfloor n^{1/2} \rfloor\}$ and for each $n \in \mathbb{N}$ the coupling $(Z'_{t,\cdot}, \tilde{Z}_{t,\cdot})$ satisfies

$$\mathbb{E} \left[\left| \beta_q^{r,s}(\mathcal{K}(n^{1/p}\{Z'_{t,1}, \dots, Z'_{t,n}\})) - \beta_q^{r,s}(\mathcal{K}(n^{1/p}\{\tilde{Z}_{t,1}, \dots, \tilde{Z}_{t,n}\})) \right| \right] \leq C\varepsilon. \tag{4.28}$$

Define the coupling time between $(Z'_{t,j})_j$ and $(\tilde{Z}_{t,j})_j$ by

$$\tau_c(t) = \inf\{j \geq t : Z'_{t,j} = \tilde{Z}_{t,j}, Z'_{t,j-1} = \tilde{Z}_{t,j-1}, \dots, Z'_{t,j-m+1} = \tilde{Z}_{t,j-m+1}\}, \tag{4.29}$$

i.e., for all $j \geq \tau_c(t)$ the chains evolve again in lockstep, viz., $Z'_{t,j} = \tilde{Z}_{t,j}$ for $j \geq \tau_c(t)$.

The coupling times τ_c admit a tail bound which involves the coefficients from the Marton coupling, viz.,

$$\begin{aligned}
\mathbb{P}(\tau_c(t) > u \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) &\leq \mathbb{P}(Z'_{t,u} \neq \tilde{Z}_{t,u} \text{ or } Z'_{t,u-1} \neq \tilde{Z}_{t,u-1} \\
&\quad \text{or } \dots \text{ or } Z'_{t,u-m+1} \neq \tilde{Z}_{t,u-m+1} \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) \\
&\leq \mathbb{P}(Z'_{t,u} \neq \tilde{Z}_{t,u} \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) + \mathbb{P}(Z'_{t,u-1} \neq \tilde{Z}_{t,u-1} \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) \\
&\quad + \dots + \mathbb{P}(Z'_{t,u-m+1} \neq \tilde{Z}_{t,u-m+1} \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) \\
&\leq \Gamma_{t,u}^{(n)} + \Gamma_{t,u-1}^{(n)} + \dots + \Gamma_{t,u-m+1}^{(n)}.
\end{aligned} \tag{4.30}$$

Since the coefficients of the Marton coupling satisfy $\sup_{n \in \mathbb{N}} \sup_{t \in \{1, \dots, n\}} \sum_{u=t}^n \Gamma_{t,u}^{(n)} < \infty$, this last calculation shows also that the expectations of the coupling times (when conditioned on $Z'_{t,t} \neq \tilde{Z}_{t,t}$) are uniformly bounded.

In the following, we write $Z'^{t,w}_{t,v}$ for the vector $(Z'_{t,v}, \dots, Z'_{t,w})$ for $v \leq w$. The notation $\tilde{Z}^{t,w}_{t,v}$ is used in the same spirit. Note that it is sufficient to focus on coupling times $\tau_c(t) \leq t + h(n)$, where $h(n) = n^{1/2}$. Indeed, as $\beta_q^{r,s}(\mathcal{K}(\emptyset)) = 0$,

$$\begin{aligned}
&n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \tilde{Z}_{t,1}^{t,t-1}, \tilde{Z}_{t,t}, \tilde{Z}_{t,t+1}^{t,n})) \mathbb{1}\{\tau_c(t) > t + h(n)\} \right] \\
&= n^{-1} \sum_{i=1}^n \mathbb{E} \left[(\beta_q^{r,s}(\mathcal{K}(n^{1/p} \tilde{Z}_{t,1}^{t,i})) - \beta_q^{r,s}(\mathcal{K}(n^{1/p} \tilde{Z}_{t,1}^{t,i-1}))) \mathbb{1}\{\tau_c(t) > t + h(n)\} \right] \\
&\leq n^{-1} \sum_{i=1}^n \sum_{j=q, q+1} \mathbb{E} \left[\mathcal{K}_j(\tilde{Z}_{t,1}^{t,i}, n^{-1/p} s; \tilde{Z}_{t,i}) \mathbb{1}\{\tau_c(t) > t + h(n)\} \right] \\
&\leq n^{-1} \sum_{i=1}^n \sum_{j=q, q+1} \mathbb{E} \left[\mathcal{K}_j(\tilde{Z}_{t,1}^{t,i}, n^{-1/p} s; \tilde{Z}_{t,i})^2 \right]^{1/2} \mathbb{P}(\tau_c(t) > t + h(n))^{1/2} \\
&\leq c_1 \mathbb{P}(\tau_c(t) > t + h(n))^{1/2},
\end{aligned} \tag{4.31}$$

where $c_1 \in \mathbb{R}_+$ does neither depend on t nor on n . The last probability is $o(1)$ uniformly in t as $n \rightarrow \infty$. Note that we use for the derivation of the last inequality that $\mathbb{E}[\mathcal{K}_j(\tilde{Z}_{t,1}^{t,i}, n^{-1/p} s; \tilde{Z}_{t,i})^2]$ is uniformly bounded, this can be shown with similar arguments which lead to (4.3) in Lemma 4.2. The same holds for $(\tilde{Z}_{t,j})_{j \geq 1}$, i.e.,

$$n^{-1} \mathbb{E} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \tilde{Z}_{t,1}^{t,t-1}, Z'_{t,t}, Z'^{t,n}_{t,t+1})) \mathbb{1}\{\tau_c(t) > t + h(n)\} \right] \leq c_1 \mathbb{P}(\tau_c(t) > t + h(n))^{1/2} \tag{4.32}$$

uniformly in t for each n .

Moreover, this last calculation also shows that the difference in (4.28) when divided by n is uniformly bounded by a constant because following the calculations in (4.31) without the indicator function, yields that both terms when divided by n are at most c_1 .

Consequently, it suffices to study only those differences in (4.28) for which the time index t is in the interval $\{1, \dots, n - \lfloor n^{1/2} \rfloor\}$. Indeed, the sum over the differences for which the time index is not in this interval is then in $\mathcal{O}(n^{1/2})$ which vanishes when divided by n .

Hence, for the rest of the proof, we consider (4.28) for $t \in \{1, \dots, n - \lfloor n^{1/2} \rfloor\}$. Also using the fact that (4.31) and (4.32) are $o(1)$, we can focus on the events where $\{\tau_c(t) \leq t + h(n)\}$

$$\begin{aligned}
&\mathbb{E} \left[\mathbb{1}\{\tau_c(t) \leq t + h(n)\} \left[\beta_q^{r,s}(\mathcal{K}(n^{1/p} \{Z'_{t,1}, \dots, Z'_{t,n}\})) - \beta_q^{r,s}(\mathcal{K}(n^{1/p} \{\tilde{Z}_{t,1}, \dots, \tilde{Z}_{t,n}\})) \right] \right] \\
&= \sum_{j=t+m-1}^{t+h(n)} \mathbb{E} \left[\mathbb{1}\{\tau_c(t) = j\} \mathbb{1}\{Z'_{t,t} \neq \tilde{Z}_{t,t}\} \left\{ \beta_q^{r,s}(\mathcal{K}(n^{1/p} \{\tilde{Z}_{t,1}^{t,t-1}, Z'_{t,t}, Z'^{t,j-1}_{t,t+1}, Z'^{t,n}_{t,j}\})) \right. \right. \\
&\quad \left. \left. - \beta_q^{r,s}(\mathcal{K}(n^{1/p} \{\tilde{Z}_{t,1}^{t,t-1}, \tilde{Z}_{t,t}, \tilde{Z}_{t,t+1}^{t,j-1}, Z'^{t,n}_{t,j}\})) \right\} \right],
\end{aligned} \tag{4.33}$$

where we use for the last equality that $\tau_c(t)$ can be at least $t + m - 1$ conditional on the event $\{Z'_{t,t} \neq \tilde{Z}_{t,t}\}$.

We study the expectations in (4.33). First note that there is a constant $c_2 \in \mathbb{R}_+$, which is independent of ε , such that

$$\begin{aligned} & \mathbb{E} \left[\beta_q^{r,s} (\mathcal{K}(n^{1/p} \{ \tilde{Z}_{t,1}^{t,t-1}, Z'_{t,t}, Z'_{t,t+1}^{t,j-1}, Z'_{t,j}^{t,n} \})) \right. \\ & \quad \left. - \beta_q^{r,s} (\mathcal{K}(n^{1/p} \{ \tilde{Z}_{t,1}^{t,t-1}, \tilde{Z}_{t,t}, \tilde{Z}_{t,t+1}^{t,j-1}, Z'_{t,j}^{t,n} \})) \mid \tau_c(t) = j, Z'_{t,t} \neq \tilde{Z}_{t,t} \right] \leq c_2(j-t). \end{aligned} \quad (4.34)$$

Indeed, both point clouds differ in at most $j - t$ points, namely, in $Z'_{t,t}^{t,j-1}$ and $\tilde{Z}_{t,t}^{t,j-1}$. Thus, we can exchange them in $j - t$ steps as in the proof of Theorem 2.6, see (4.10) and (4.11). Afterwards, we can apply the geometric lemma to bound the single terms, see also the proof of Theorem 2.6 in (4.12), and we end up with the following bound for (4.34)

$$\begin{aligned} & \sum_{\ell=1}^{j-t} \sum_{k=q,q+1} \mathbb{E} \left[\mathcal{K}_k \left(\tilde{Z}_{t,1}^{t,t-1+\ell} \cup Z'_{t,t-1+\ell}^n, n^{-1/p} s; Z'_{t,t+\ell-1} \right) \mid \tau_c(t) = j, Z'_{t,t} \neq \tilde{Z}_{t,t} \right] \\ & \quad + \sum_{\ell=1}^{j-t} \sum_{k=q,q+1} \mathbb{E} \left[\mathcal{K}_k \left(\tilde{Z}_{t,1}^{t,t-1+\ell} \cup Z'_{t,t-1+\ell}^n, n^{-1/p} s; \tilde{Z}_{t,t+\ell-1} \right) \mid \tau_c(t) = j, Z'_{t,t} \neq \tilde{Z}_{t,t} \right]. \end{aligned} \quad (4.35)$$

Now, we can conclude from (4.3) in Lemma 4.2 that each summand is bounded by a constant $c_2 \in \mathbb{R}_+$ a.s. This constant is independent of ε and uniform in ℓ, j, t and n because there is a minimum of uncertainty in each transition step. Hence, (4.35) is bounded by $c_2(j - t)$. This yields then (4.34).

Furthermore, note that by (4.30)

$$\begin{aligned} & \sum_{j=t}^{t+h(n)} (j-t) \mathbb{P}(\tau_c(t) = j \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) \leq \mathbb{E} \left[\tau_c(t) - t \mid Z'_{t,t} \neq \tilde{Z}_{t,t} \right] \\ & \leq \sum_{j=0}^{\infty} \mathbb{P}(\tau_c(t) - t > j \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) \\ & = \sum_{j=t+m-1}^{\infty} \mathbb{P}(\tau_c(t) > j \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) \\ & \leq \sum_{j=t+m-1}^{\infty} \Gamma_{t,j}^{(n)} + \dots + \Gamma_{t,j-m+1}^{(n)}, \end{aligned}$$

which is uniformly bounded in t and n , see (4.21).

This means that (4.33) is at most (times the constant c_2 from (4.34))

$$\begin{aligned} & \sum_{j=t}^{t+h(n)} (j-t) \mathbb{E} \left[\mathbb{1}\{\tau_c(t) = j\} \mathbb{1}\{Z'_{t,t} \neq \tilde{Z}_{t,t}\} \right] \leq \mathbb{E} \left[\mathbb{1}\{Z'_{t,t} \neq \tilde{Z}_{t,t}\} \sum_{j=t+m-1}^{t+h(n)} (j-t) \mathbb{P}(\tau_c(t) \mid Z'_{t,t} \neq \tilde{Z}_{t,t}) \right] \\ & \leq c_3 \mathbb{P}(Z'_{t,t} \neq \tilde{Z}_{t,t}) \leq c_3 \varepsilon, \end{aligned}$$

for a constant $c_3 \in \mathbb{R}_+$. Combining these results yields (4.28). This completes the proof. \square

Proof of Corollary 2.9. It is shown in Proposition 3.4 in Hiraoka et al. [2018] that the pointwise convergence of persistent Betti numbers implies the vague convergence of the persistent diagram. \square

Proof of Theorem 3.1. The statement follows immediately from Theorem 2.7. \square

Proof of Theorem 3.2. The proof is very similar to that of Theorem 2.8. To this end, we only study the main differences in detail. First, we construct an ε -approximation \tilde{X} of X . We consider the joint distribution of $\{X_u : u \leq (1, \dots, 1)\}$,

which is completely determined by the joint density $g: [0, 1]^{2^d p} \rightarrow (0, \infty)$. Let $\varepsilon > 0$ and choose a discrete approximation $g_\varepsilon: [0, 1]^{2^d p} \rightarrow (0, \infty)$ of g such that the conditional densities $f_{\varepsilon, s}$ derived from g_ε are strictly positive and satisfy

$$|f_{\varepsilon, s}(x|y) - f_s(x|y)| \leq \varepsilon, \quad \forall x \in [0, 1]^p, \quad \forall y \in [0, 1]^{\|s\|_1 p}, \quad \forall s \in \{0, 1\}^d.$$

(This is the analog requirement to (4.20)). Obviously, the discrete (conditional) densities $f_{\varepsilon, s}$ determine the random field \tilde{X} completely. Also, due to the blocked structure of the densities $f_{\varepsilon, s}$ and the condition from (3.9), the random field \tilde{X} satisfies the requirements of Theorem 3.1. So it is sufficient to study the difference

$$\pi(N)^{-1} \mathbb{E} \left[\beta_q^{r, s}(\mathcal{K}(\pi(N)^{1/p} \mathbb{X}_N)) \right] - \pi(N)^{-1} \mathbb{E} \left[\beta_q^{r, s}(\mathcal{K}(\pi(N)^{1/p} \tilde{\mathbb{X}}_N)) \right] \quad (4.36)$$

for an arbitrary but fixed $N \in \mathbb{N}_+^d$.

We use the same expansion for this difference as in the case of Markov chains, see (4.25), (4.26) and (4.27). But this time we use the ordering $>_d$ to for the expansion. We obtain for each $u \in \mathbb{N}_+^d$ a coupling $((Z'_{u, v}, \tilde{Z}_{u, v}) : v \in \mathbb{N}_+^d)$ with the properties

- (i) $Z'_{u, w} = \tilde{Z}_{u, w}$, $\forall w <_d u$, which are distributed according to the $f_{\varepsilon, s}$, $s \in \{0, 1\}^d$,
- (ii) $\mathbb{P}(Z'_{u, u} \neq \tilde{Z}_{u, u} | Z'_{u, w} = \tilde{Z}_{u, w}, \forall w <_d u) \leq 2\varepsilon$ a.s.,
- (iii) $Z'_{u, u}$ is distributed according to the f_s , $s \in \{0, 1\}^d$,
- (iv) $\tilde{Z}_{u, u}$ is distributed according to the $f_{\varepsilon, s}$, $s \in \{0, 1\}^d$,
- (v) $\mathbb{P}(Z'_{u, v} \neq \tilde{Z}_{u, v} | Z'_{u, w} = \tilde{Z}_{u, w}, w <_d u, Z'_{u, u}, \tilde{Z}_{u, u}) \leq \Gamma_{u, v}^{(\infty)}$ a.s., $\forall v >_d u$,
- (vi) and the $Z'_{u, v}, \tilde{Z}_{u, v}$ are distributed according to the f_s , $s \in \{0, 1\}^d$, for all $v >_d u$.

Consequently, using that $\tilde{Z}_{u, v} = Z'_{u, v}$, for all $v <_d u$, we can write the difference in (4.36) as

$$\pi(N)^{-1} \sum_{\substack{u \in \mathbb{N}_+^d : u \leq N \\ u <_d N}} \mathbb{E} \left[\left(\beta_q^{r, s}(\mathcal{K}(\pi(N)^{1/p} \{Z'_{u, v} : v \leq N\})) - \beta_q^{r, s}(\mathcal{K}(\pi(N)^{1/p} \{\tilde{Z}_{u, v} : v \leq N\})) \right) \right]. \quad (4.37)$$

For a coupling $(Z'_{u, \cdot}, \tilde{Z}_{u, \cdot})$, we define the coupling time

$$\Delta\tau_c(u) = \inf \{k \geq 0 \mid Z'_{u, v} = \tilde{Z}_{u, v}, \forall v \in \mathbb{N}^d \text{ such that } \|u - v\|_{max} = k \text{ and } v \geq u\}. \quad (4.38)$$

So $\Delta\tau_c(u)$ is determined by the causal dependence pattern which is derived from the factorization of the joint distribution according to the ordering $>_d$. Note that both random fields $Z'_{u, \cdot}$ and $\tilde{Z}_{u, \cdot}$ move in lockstep after $\Delta\tau_c(u)$.

Consider the tail of the coupling time $\Delta\tau_c$ at location u

$$\begin{aligned} & \mathbb{P}(\Delta\tau_c(u) > k \mid Z'_{u, u} \neq \tilde{Z}_{u, u}) \\ &= \mathbb{P}(Z'_{u, v} \neq \tilde{Z}_{u, v} \text{ for one } v \geq u \text{ with } \|v - u\|_{max} = k \mid Z'_{u, u} \neq \tilde{Z}_{u, u}) \\ &\leq \sum_{\substack{v: v \geq u \\ \|v - u\|_{max} = k}} \mathbb{P}(Z'_{u, v} \neq \tilde{Z}_{u, v} \text{ for one } v \geq u \text{ with } \|v - u\|_{max} = k \mid Z'_{u, u} \neq \tilde{Z}_{u, u}) \\ &\leq \sum_{\substack{v: v \geq u \\ \|v - u\|_{max} = k}} \Gamma_{u, v}^{(\infty)} \\ &\leq c_1 k^{d-1} \sup_{\substack{v: v \geq u \\ \|v - u\|_{max} = k}} \Gamma_{u, v}^{(\infty)} = o(1), \quad k \rightarrow \infty, \end{aligned}$$

by condition (3.9) for a constant $c_1 \in \mathbb{R}_+$ which is independent of u, k, N . Moreover, the condition

$$\min\{N_i : i = 1, \dots, d\} / \max\{N_i : i = 1, \dots, d\} \geq \bar{c}, \quad (4.39)$$

for some constant $\bar{c} \in \mathbb{R}_+$, allows us to consider only these u which satisfy $\|u\|_{max} \leq \|N\|_{max} - h(N)$, where this time $h(N) = \pi(N)^{1/(2d)}$. Indeed, there is a constant $c_2 \in \mathbb{R}_+$, which does not depend on u and N , such that the number of locations u which satisfy $\|u\|_{max} \geq \|N\|_{max} - h(N)$ and $u \leq N$ is at most

$$c_2 \|N\|_{max}^{d-1} h(N) = c_2 \|N\|_{max}^{d-1} \pi(N)^{1/(2d)} \leq c_2 \|N\|_{max}^{d-1+1/2} = o(\pi(N)), \quad N \rightarrow \infty,$$

because of the condition (4.39). Consequently, carrying out the same argument as in the proof of Theorem 2.8 shows that we can neglect observations at locations u which satisfy $\|u\|_{max} \geq \|N\|_{max} - h(N)$ and $u \leq N$.

Moreover, we obtain as in (4.31)

$$\begin{aligned} & \pi(N)^{-1} \mathbb{E} \left[\beta_q^{r,s} (\mathcal{K}(\pi(N))^{1/p} \{\tilde{Z}_{u,v} : v \leq N\}) \mathbf{1}\{\Delta\tau_c(u) > h(N)\} \right] \\ & + \pi(N)^{-1} \mathbb{E} \left[\beta_q^{r,s} (\mathcal{K}(\pi(N))^{1/p} \{Z'_{u,v} : v \leq N\}) \mathbf{1}\{\Delta\tau_c(u) > h(N)\} \right] \\ & \leq c_3 \mathbb{P}(\Delta\tau_c(u) > h(N))^{1/2} = o(1), \quad N \rightarrow \infty, \end{aligned}$$

where the constant c_3 is independent of the location u and N .

This means that we can focus on the $u \in \{w \in \mathbb{N}_+^d \mid w \leq N, \|w\|_{max} \leq \|N\|_{max} - h(N)\}$ and can additionally condition on the event $\{\Delta\tau_c(u) \leq h(N)\}$ for the difference in (4.37). Under this restriction we obtain from the definition of the coupling time $\Delta\tau_c(u)$ and conditional on the event $\{Z'_{u,u} \neq \tilde{Z}_{u,u}\}$

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}\{\Delta\tau_c(u) \leq h(N)\} \left(\beta_q^{r,s} (\mathcal{K}(\pi(N))^{1/p} \{Z'_{u,v} : v \leq N\}) \right. \right. \\ & \quad \left. \left. - \beta_q^{r,s} (\mathcal{K}(\pi(N))^{1/p} \{\tilde{Z}_{u,v} : v \leq N\}) \right) \mid Z'_{u,u} \neq \tilde{Z}_{u,u} \right] \\ & \leq c_4 \sum_{k=0}^{h(N)} k^d \mathbb{P}(\Delta\tau_c(u) = k \mid Z'_{u,u} \neq \tilde{Z}_{u,u}) \\ & \leq c_4 \mathbb{E} \left[\Delta\tau_c(u)^d \mid Z'_{u,u} \neq \tilde{Z}_{u,u} \right] \\ & \leq c_5 \sum_{k=0}^{\infty} k^{d-1} \mathbb{P}(\Delta\tau_c(u) > k \mid Z'_{u,u} \neq \tilde{Z}_{u,u}) \\ & \leq c_5 \sum_{k=0}^{\infty} k^{2(d-1)} \sup_{\substack{v: v \geq u \\ \|v-u\|_{max}=k}} \Gamma_{u,v}^{(\infty)} \leq c_6 \end{aligned}$$

uniformly in u and uniformly in N for three constants c_4, \dots, c_6 , using the condition from (3.9).

This shows that each expectation in (4.37) is at most $2c_6\varepsilon$, using $\mathbb{P}(Z'_{u,u} \neq \tilde{Z}_{u,u} \mid Z'_{u,w} = \tilde{Z}_{u,w}, \forall w <_d u) \leq 2\varepsilon$. Consequently, (4.37) and (4.36) are at most $2c_6\varepsilon$. This completes the proof of Theorem 3.2. \square

A McDiarmid inequalities for Marton couplings

In this section we study McDiarmid inequalities for Marton couplings. Notable contributions to this topic are Samson [2000], Chazottes et al. [2007], Kontorovich and Ramanan [2008], Redig and Chazottes [2009]. We shall first state a result of Paulin [2015].

Definition A.1 (Partition). A partition of a random vector $Z = (Z_1, \dots, Z_N)$ is a deterministic division of Z into random variables \hat{Z}_i , $i = 1, \dots, n$, for some $n \leq N$ such that the set $\{Z_1, \dots, Z_N\}$ is partitioned by $(\hat{Z}_i)_{i=1, \dots, n}$. Denote the number of elements of \hat{Z}_i by $s(\hat{Z}_i)$ and write $s(\hat{Z})$ for the size of the partition which is $\max_{i=1, \dots, n} s(\hat{Z}_i)$.

Theorem A.2 (McDiarmid's inequality, Paulin [2015]). Let $Z = (Z_1, \dots, Z_N)$ be a random variable in $\Lambda = \Lambda_1 \times \dots \times \Lambda_N$. Assume that Z admits a partitioning $\hat{Z} = (\hat{Z}_1, \dots, \hat{Z}_n)$ which allows a Marton coupling with mixing matrix $\Gamma \in \mathbb{R}^{n \times n}$. Let $\varphi : \Lambda \rightarrow \mathbb{R}$ be Lipschitz continuous w.r.t. the Hamming distance, i.e., there is a $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ such that

$$|\varphi(x) - \varphi(y)| \leq \sum_{j=1}^n c_j \mathbf{1}\{x_j \neq y_j\}, \quad x, y \in \Lambda. \quad (\text{A.1})$$

Set $\mathcal{J}_i := \{j = 1, \dots, N : Z_j \in \hat{Z}_i\}$ and $C_i(c) := \sum_{j \in \mathcal{J}_i} c_j$ for $i = 1, \dots, n$. Then

$$\log \mathbb{E} [\exp (\gamma (\varphi(Z) - \mathbb{E} [\varphi(Z)]))] \leq \gamma^2 \|\Gamma C(c)\|^2 / 8. \quad (\text{A.2})$$

In particular,

$$\mathbb{P} (|\varphi(Z) - \mathbb{E} [\varphi(Z)]| \geq t) \leq 2 \exp \left(-\frac{2t^2}{\|\Gamma C(c)\|^2} \right) \leq 2 \exp \left(-\frac{2t^2}{\|\Gamma\|^2 \|c\|^2 s(\hat{Z})} \right). \quad (\text{A.3})$$

Proof of Theorem A.2. The proof uses the following lemma of Devroye and Lugosi [2012]: Let \mathcal{F} be a sub- σ -algebra, U, V, W random variables which satisfy $U \leq V \leq W$ a.s. Moreover, U, W are \mathcal{F} -measurable and $\mathbb{E} [V | \mathcal{F}] = 0$. Then

$$\log \mathbb{E} [\exp (\gamma V) | \mathcal{F}] \leq \gamma^2 (U - W)^2 / 8.$$

Consider the natural filtration of the random vector \hat{Z} , i.e., $\mathcal{F}_i = \sigma(\hat{Z}_1, \dots, \hat{Z}_i)$ for $i = 0, \dots, n$ and define $\hat{\varphi}(\hat{Z}) := \varphi(Z)$. Then $\hat{\varphi}$ is also Lipschitz continuous w.r.t. Hamming distance, more precisely,

$$|\hat{\varphi}(x) - \hat{\varphi}(y)| \leq \sum_{j=1}^n c_j \mathbf{1}\{x_j \neq y_j\} \leq \sum_{i=1}^n C_i(c) \mathbf{1}\{\hat{x}_i \neq \hat{y}_i\}.$$

Set $V_i = \mathbb{E} [\hat{\varphi}(\hat{Z}) | \mathcal{F}_i] - \mathbb{E} [\hat{\varphi}(\hat{Z}) | \mathcal{F}_{i-1}]$ for $i = 1, \dots, n$. Moreover, define for $a \in \Lambda_i$

$$I_i(a) := \int_{\Lambda_{i+1} \times \dots \times \Lambda_n} \mathbb{P} \left(\hat{Z}_{i+1} \in d\hat{z}_{i+1}, \dots, \hat{Z}_n \in d\hat{z}_n \mid \hat{Z}_1, \dots, \hat{Z}_{i-1}, \hat{Z}_i = a \right) \hat{\varphi} \left(\hat{Z}_1, \dots, \hat{Z}_{i-1}, a, \hat{z}_{i+1}, \dots, \hat{z}_n \right).$$

And write ν_i for the conditional distribution of \hat{Z}_i given $(\hat{Z}_1, \dots, \hat{Z}_{i-1})$, i.e.,

$$\nu_i(\cdot) = \mathbb{M}_{\hat{Z}_i | (\hat{Z}_1, \dots, \hat{Z}_{i-1})} \left((\hat{Z}_1, \dots, \hat{Z}_{i-1}), \cdot \right).$$

Then, noting that the measure ν_i is random and with some elementary calculations, it follows that

$$V_i \leq \operatorname{ess\,sup}_{\text{w.r.t. } \nu_i} I_i(\cdot) - \operatorname{ess\,inf}_{\text{w.r.t. } \nu_i} I_i(\cdot) \quad \text{a.s.}$$

Now let $\varepsilon > 0$ be arbitrary but fixed. Choose $a^*, b^* \in \Lambda_i$ such that $I_i(a^*) \geq \operatorname{ess\,sup}_{\text{w.r.t. } \nu_i} I_i(\cdot) - \frac{\varepsilon}{2}$ and $I_i(b^*) \leq$

$\text{ess inf}_{\text{w.r.t. } \nu_i} I_i(\cdot) + \frac{\varepsilon}{2}$. Next, we use the Marton coupling of \hat{Z} to obtain

$$\begin{aligned} I_i(a^*) - I_i(b^*) &= \mathbb{E} \left[\hat{\varphi} \left(\hat{Z}^{(\hat{Z}_1, \dots, \hat{Z}_{i-1}, a^*, b^*)} \right) - \hat{\varphi} \left(\hat{Z}'^{(\hat{Z}_1, \dots, \hat{Z}_{i-1}, a^*, b^*)} \right) \mid \hat{Z}_1, \dots, \hat{Z}_{i-1} \right] \\ &\leq \sum_{k=i}^n C_k(c) \mathbb{P} \left(\hat{Z}_k^{(\hat{Z}_1, \dots, \hat{Z}_{i-1}, a^*, b^*)} \neq \hat{Z}'_k^{(\hat{Z}_1, \dots, \hat{Z}_{i-1}, a^*, b^*)} \mid \hat{Z}_1, \dots, \hat{Z}_{i-1} \right) \\ &\leq \sum_{k=i}^n C_k(c) \Gamma_{i,k}. \end{aligned}$$

And as ε was arbitrary,

$$\text{ess sup}_{\text{w.r.t. } \nu_i} I_i(\cdot) - \text{ess inf}_{\text{w.r.t. } \nu_i} I_i(\cdot) \leq \sum_{k=i}^n C_k(c) \Gamma_{i,k}.$$

Moreover, we have

$$\text{ess inf}_{\text{w.r.t. } \nu_i} I_i(\cdot) - \mathbb{E} \left[\hat{\varphi}(\hat{Z}) \mid \mathcal{F}_{i-1} \right] \leq V_i \leq \text{ess sup}_{\text{w.r.t. } \nu_i} I_i(\cdot) - \mathbb{E} \left[\hat{\varphi}(\hat{Z}) \mid \mathcal{F}_{i-1} \right], \quad a.s.$$

and both the left- and the right-hand-side are \mathcal{F}_{i-1} -measurable. Consequently, using the lemma of Devroye and Lugosi [2012], we find that

$$\mathbb{E} \left[\exp(\gamma V_i) \mid \mathcal{F}_{i-1} \right] \leq \exp \left(\frac{\gamma^2}{8} \left(\sum_{k=i}^n C_k(c) \Gamma_{i,k} \right)^2 \right).$$

This establishes the claim in (A.2). A standard computation yields (A.3), note that the second inequality follows from the inequalities $\|\Gamma C(c)\|^2 \leq \|\Gamma\|^2 \|C(c)\|^2 \leq \|\Gamma\|^2 \|c\|^2 s(\hat{Z})$. □

The next proposition is due to Fiebig [1993] and a consequence of Goldstein's maximal coupling, Goldstein [1979]. See also Paulin [2015] Proposition 2.6 and Samson [2000] Proposition 2.

Proposition A.3 (Fiebig [1993], p. 482, (2.1)). *Let P and Q be two probability distributions on some common Polish space $\Lambda_1 \times \dots \times \Lambda_N$ both admitting a strictly positive density w.r.t. to a measure ρ . Then there is a coupling of random vectors $X = (X_1, \dots, X_N)$, $Y = (Y_1, \dots, Y_N)$ such that $\mathcal{L}(X) = P$, $\mathcal{L}(Y) = Q$ and*

$$\mathbb{P}(X_i \neq Y_i) \leq d_{TV}(\mathcal{L}(X_i, \dots, X_N), \mathcal{L}(Y_i, \dots, Y_N)), \quad i = 1, \dots, N.$$

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