On the Distribution of SINR for the MMSE MIMO Receiver and Performance Analysis

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***Abstract*— This paper studies the statistical distribution of the signal-to-interference-plus-noise ratio (SINR) for the minimum mean square error (MMSE) receiver in multiple-input-multiple-output (MIMO) wireless communications. The channel model is assumed to be (transmit) correlated Rayleigh flat-fading with unequal powers. The SINR can be decomposed into two independent random variables: SINR $=$ SINR$^{\text{ZF}} + T$, where SINR$^{\text{ZF}}$ corresponds to the SINR for a zero-forcing (ZF) receiver and has an exact Gamma distribution. This paper focuses on characterizing the statistical properties of T using the results from random matrix theory. First three asymptotic moments of T are derived for uncorrelated channels and channels with equicorrelations. For general correlated channels, some limiting upper bounds for the first three moments are also provided. For uncorrelated channels and correlated channels satisfying certain conditions, it is proved that T converges to a Normal random variable. A Gamma distribution and a generalized Gamma distribution are proposed as approximations to the finite sample distribution of T. Simulations suggest that these approximate distributions can be used to accurately estimate the probability of errors even for very small dimensions (e.g., 2 transmit antennas).**

***Index Terms*— asymptotic distributions, channel correlation, error probability, Gamma approximation, minimum mean square error receiver, multiple-input-multiple-output system, random matrix, signal-to-interference-plus-noise ratio,**

I. Introduction

This study considers the following signal and channel model in a multiple-input-multiple-output (MIMO) system,

$$y_r = \frac{1}{\sqrt{m}}\mathbf{H_W}\mathbf{R_t}^{\frac{1}{2}}\mathbf{P}^{\frac{1}{2}}x_t + n_c = \frac{1}{\sqrt{m}}\mathbf{H}x_t + n_c, \quad (1)$$

where $x_t \in \mathbb{C}^p$ is the (normalized) transmitted signal vector and $y_r \in \mathbb{C}^m$ is the received signal vector. Here p is the number of transmit antennas and m is the number of receive antennas. $\mathbf{H_W} \in \mathbb{C}^{m\times p}$ consists of i.i.d. standard complex Normal entries. $\mathbf{R_t} \in \mathbb{C}^{p\times p}$ is the transmitter correlation matrix. $\mathbf{P} = \text{diag}[\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_p] \in \mathbb{R}^{p\times p}$; $\tilde{c}_k = c_k\frac{m}{p}$, where c_k is the signal-to-noise ratio (SNR) for the k^{th} spatial stream. This definition of SNR is consistent with [1] (Section 7.4). $\mathbf{H} = \mathbf{H_W}\mathbf{R_t}^{\frac{1}{2}}\mathbf{P}^{\frac{1}{2}} \in \mathbb{C}^{m\times p}$ is treated as the channel matrix. $n_c \in \mathbb{C}^m$ is the complex noise vector and is assumed to have zero mean and identity covariance. Note that the power matrix $\mathbf{P}$ has terms involving the variance of the noise. The correlation matrix $\mathbf{R}_t$ and power matrix $\mathbf{P}$ are assumed to be non-random. Also, we restrict our attention to $p \leq m$.

We consider the popular linear minimum mean square error (MMSE) receiver. Conditional on the channel matrix $\mathbf{H}$, the signal-to-interference-plus-noise ratio (SINR) on the k^{th} spatial stream can be expressed as (e.g., [1]–[6])

$$\text{SINR}_k = \frac{1}{\text{MMSE}_k} - 1 = \frac{1}{\left[\left(\mathbf{I}_p + \frac{1}{m}\mathbf{H}^\dagger\mathbf{H}\right)^{-1}\right]_{kk}} - 1, \quad (2)$$

where $\mathbf{I}_p$ is a $p\times p$ identity matrix, and $\mathbf{H}^\dagger$ is the Hermitian transpose of $\mathbf{H}$. Note that (2), in the same form as (7.49) of [1], is derived based on the second order statistics of the input signals, not restricted to binary signals.

For binary inputs, Verdu [4] (6.47) provides the exact formula for computing the bit error rate (BER) (also see [7]). Conditional on $\mathbf{H}$, this BER formula requires computing 2^{p-1} Q-functions. To compute BER unconditionally, we need to sample $\mathbf{H}$ enough times (e.g., 10^5) to get a reliable estimate. When $p \geq 32$ (or $p \geq 64$), the computations become intractable [4], [8].

Recently, study of the asymptotic properties of multiuser receivers (e.g., [2]–[4], [6], [8]–[11]) has received a lot of attention. Works that relate directly to the content of this paper include Tse and Hanly [11] and Verdu and Shamai [6], who independently derived the asymptotic first moment of SINR for uncorrelated channels. Tse and Zeitouni [3] proved the asymptotic Normality of SINR for the equal power case, and commented on the possibility of extending the result to the unequal powers scenario. Zhang *et. al.* [12] proved the asymptotic Normality of the multiple access interference (MAI), which is closely related to SINR. Guo *et. al.* [8] proved the asymptotic Normality of the decision statistics for a variety of linear multiuser receivers. [8] considered a general power distribution and corresponding unconditional asymptotic behavior.

Based on the asymptotic Normality results, Poor and Verdu [2] (also in [4], [8]) proposed using the limiting BER (denoted by BER$_\infty$) for binary modulations, which is a single Q-function,

$$\text{BER}_\infty = Q\left(\sqrt{\text{E}(\text{SINR}_k)_\infty}\right) = \int_{\sqrt{\text{E}(\text{SINR}_k)_\infty}}^{\infty} e^{-t^2/2}dt, \quad (3)$$

where $\text{E}(\text{SINR}_k)_\infty$ denotes the asymptotic first moment of SINR$_k$.

Equation (3) is convenient and accurate for large dimensions. However, its accuracy for small dimensions is of some concern. For instance, [8] compared the asymptotic BER with

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simulation results, which showed that even with $p = 64$ there existed significant discrepancies. In general (3) will underestimate the true BER. For example, in our simulations, when $m = 16$, $p = 8$, SNR = 15 dB, the asymptotic BER given by (3) is roughly $\frac{1}{10000}$ of the exact BER. In current practice, CDMA channels with m, p between 32 and 64 are typical and in multi-antenna systems arrays of 4 antennas are typical but arrays with 8 to 16 antennas would be feasible in the near future [9]. Therefore it would be useful if one can compute error probabilities both efficiently and accurately. Another motivation to have accurate and simple BER expressions comes from system optimization designs [13], [14].

This study addresses three aspects related to improving the accuracy in computing probability of errors. First, the known formulae for the asymptotic moments can be improved at small dimensions. Secondly, while the asymptotic BER (3) converges very slowly [4] (page 305), the accuracy can be improved by considering other asymptotically equivalent distributions using higher moments. Thirdly, the presence of channel correlations may (seriously) affect the moments and BER. In fact, when the dimension increases, the effect of correlation tends to invalidate the independence assumption [15]. To the best of our knowledge, the asymptotic SINR results in the existing literature do not take into account correlations. Müller [16] has commented that the presence of correlations makes the analysis very difficult .

The channel model in (1) takes into account the (transmit) channel correlations. This model does not consider the receiver correlation and assumes a Gaussian channel (Rayleigh flat-fading), while [3] and [8] did not assume Gaussianity. Removing the Normality assumption will still result in the same first moment of SINR but different second moment (see [3]). Ignoring the channel correlation, however, may produce quite different results even for the first moment. For example, it will be shown later that, with equal power and constant correlation ρ (equicorrelation model), the first moment is only about $(1-\rho)$ times the first moment without correlation.

Our work starts with a crucial observation: assuming a correlated channel model in (1), the SINR can be decomposed into two independent components

$$\text{SINR}_k = \text{SINR}_k^{\text{ZF}} + T_k = S_k + T_k, \tag{4}$$

where $\text{SINR}_k^{\text{ZF}}$, denoted by S_k, is the corresponding SINR for the zero-forcing (ZF) receiver, which, conditional on $\mathbf{H}$, can be expressed as (e.g. [1], [3], [17]),

$$\text{SINR}_k^{\text{ZF}} = \frac{1}{\left[\left(\frac{1}{m}\mathbf{H}^\dagger\mathbf{H}\right)^{-1}\right]_{kk}}. \tag{5}$$

It will be shown that S_k has a Gamma distribution. Because S_k and T_k are independent, we can focus only on T_k. As S_k is often the dominating component, separating out S_k is expected to improve the accuracy of the approximation. For uncorrelated channels and channels with equicorrelation, we derive the first three asymptotic moments for T_k (thus for SINR_k also), which match the simulations remarkably well for finite dimensions. For general correlated channels, some limiting upper bounds for the first three moments are proposed.

We prove the asymptotic Normality of T_k for correlated channels under certain conditions. The proof of asymptotic Normality provides a rigorous basis for proposing approximate distributions. Since T_k is proved to converge to a non-random distribution (i.e., a Normal with non-random mean and variance), the same asymptotic results hold for the "unconditional" SINR_k, by dominated convergence. The subtle difference between conditional and unconditional SINR_k becomes important if one would like to extend to random power distributions or random correlations (see [8]).

As an alternative to (3), a well-known approximation to the true BER for binary modulations would be

$$\text{BER}_a \triangleq \int_0^\infty \left(\int_{\sqrt{x}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\right) dF_{\text{SINR}_k}(x), \tag{6}$$

where F_{SINR_k} is the cumulative distribution function (CDF) of SINR_k. (6) converges to (3) as long as the asymptotic Normality holds. We propose using a Gamma and a generalized Gamma distribution to approximate F_{SINR_k}. In particular, if T_k is approximated as a generalized Gamma random variable, combining the exact distribution of S_k, we are able to produce very accurate BER curves even for $m = 4$.

We will use binary signals as a test case for evaluating our methods. We assume BFSK for obtaining the same limiting BER (3) as in [2], [4], [8]. It should be clear that our methodology applies to other constellations such as M-QAM.

The paper is organized as follows. Section II describes how to decompose $\text{SINR}_k = \text{SINR}_k^{\text{ZF}} + T_k$. Section III has the derivation of the exact distribution of $\text{SINR}_k^{\text{ZF}}$ and its independence from T_k. Section IV contains the derivation of the first three asymptotic moments (or upper bounds on moments) of T_k. Asymptotic Normality of T_k is proved in Section V. In Section VI the Gamma and generalized Gamma distribution approximations are proposed. Section VII has a demonstration about how to apply our results on SINR in computing the probability of errors. Section VIII compares our theoretical results with Monte Carlo simulations.

II. Decomposition of SINR_k

Let $\mathbf{A} = \mathbf{I}_p + \frac{1}{m}\mathbf{H}^\dagger\mathbf{H}$. Apply the following shifting operation,

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} a_1 & a_2 & \dots & a_k & \dots & a_p \end{pmatrix} \\ &\overset{\text{shift}}{\longrightarrow} \begin{pmatrix} \mathbf{A}_{kk} & a_{k(-k)}^\dagger \\ a_{k(-k)} & \mathbf{A}_{(-k,-k)} \end{pmatrix} = \tilde{\mathbf{A}},\end{aligned} \tag{7}$$

where $a_k \in \mathbb{C}^{p\times 1}$ stands for the kth column of A, $a_{k(-k)} \in \mathbb{C}^{(p-1)\times 1}$ is a_k with the kth entry removed. $\mathbf{A}_{(-k,-k)} \in \mathbb{C}^{(p-1)\times(p-1)}$ is $\mathbf{A}$ with the kth column and kth row removed. Then,

$$\begin{aligned}\left[\mathbf{A}^{-1}\right]_{kk} &= \left[\tilde{\mathbf{A}}^{-1}\right]_{11} \\ &= \left(\mathbf{A}_{kk} - a_{k(-k)}^\dagger \left(\mathbf{A}_{(-k,-k)}\right)^{-1} a_{k(-k)}\right)^{-1}.\end{aligned} \tag{8}$$

From (8), it follows that (2) can be expressed as

$$\begin{aligned}
&\mathrm{SINR}_k \\
=&\frac{1}{\left(\mathbf{A}_{kk} - a_{k(-k)}^\dagger \left(\mathbf{A}_{(-k,-k)}\right)^{-1} a_{k(-k)}\right)^{-1}} - 1 \\
=&\frac{1}{m} h_k^\dagger h_k - \\
&\frac{1}{m^2} h_k^\dagger \mathbf{H}_{(-k)} \left(\mathbf{I}_{p-1} + \frac{1}{m}\mathbf{H}_{(-k)}^\dagger \mathbf{H}_{(-k)}\right)^{-1} \mathbf{H}_{(-k)}^\dagger h_k, \quad (9)
\end{aligned}$$

where $h_k \in \mathbb{C}^{m\times 1}$ stands for the kth column of $\mathbf{H}$, $\mathbf{H}_{(-k)} \in \mathbb{C}^{m\times(p-1)}$ is $\mathbf{H}$ with the kth column removed.

Consider the singular value decomposition (SVD): $\mathbf{H}_{(-k)} = \mathbf{U}\mathbf{D}\mathbf{V}^\dagger$, $\mathbf{U} \in \mathbb{C}^{m\times(p-1)}$, $\mathbf{D} \in \mathbb{R}^{(p-1)\times(p-1)}$, $\mathbf{V} \in \mathbb{C}^{(p-1)\times(p-1)}$, $\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}_{p-1}$, $\mathbf{V}^\dagger V = \mathbf{V}\mathbf{V}^\dagger = \mathbf{I}_{p-1}$. Let $\mathbf{U}_c$ be the orthogonal complement of $\mathbf{U}$, implying that $\mathbf{U}^\dagger\mathbf{U}_c = \mathbf{0}_{(p-1)\times(m-p+1)}$, and $\tilde{\mathbf{U}} = [\mathbf{U}\ \mathbf{U}_c]$ satisfies $\tilde{\mathbf{U}}^\dagger\tilde{\mathbf{U}} = \tilde{\mathbf{U}}\tilde{\mathbf{U}}^\dagger = \mathbf{I}_m$. Plug these into (9) to get

$$\begin{aligned}
\mathrm{SINR}_k =& \frac{1}{m} h_k^\dagger \tilde{\mathbf{U}}\tilde{\mathbf{U}}^\dagger h_k \\
&- \frac{1}{m^2} h_k^\dagger \mathbf{U}\mathbf{D}^2 \left(\mathbf{I}_{p-1} + \frac{1}{m}\mathbf{D}^2\right)^{-1} \mathbf{U}^\dagger h_k. \quad (10)
\end{aligned}$$

Now, expand $\tilde{\mathbf{U}}^\dagger h_k = \begin{bmatrix} \mathbf{U}^\dagger h_k \\ \mathbf{U}_c^\dagger h_k \end{bmatrix} = \begin{bmatrix} t_k \\ s_k \end{bmatrix}$, $t_k \in \mathbb{C}^{(p-1)\times 1}$, $s_k \in \mathbb{C}^{(m-p+1)\times 1}$. Then (10) becomes

$$\begin{aligned}
\mathrm{SINR}_k &= \frac{1}{m} s_k^\dagger s_k + \frac{1}{m} t_k^\dagger \left(\mathbf{I}_{p-1} + \frac{1}{m}\mathbf{D}^2\right)^{-1} t_k \\
&= \frac{1}{m}\sum_{i=1}^{m-p+1} \|s_{k,i}\|^2 + \frac{1}{m}\sum_{i=1}^{p-1} \frac{\|t_{k,i}\|^2}{1+\frac{1}{m}d_i^2} \\
&= S_k + T_k, \quad (11)
\end{aligned}$$

where d_i is the ith diagonal entry of $\mathbf{D}$.

The same shifting and SVD operations on $\mathrm{SINR}_k^{\mathrm{ZF}}$ in (5) yields

$$\mathrm{SINR}_k^{\mathrm{ZF}} = \frac{1}{m} s_k^\dagger s_k = S_k. \quad (12)$$

This proves the decomposition $\mathrm{SINR}_k = \mathrm{SINR}_k^{\mathrm{ZF}} + T_k$. So far, no statistical properties of $\mathbf{H}$ are used, hence the decomposition is not restricted to Gaussian channels.

In the case $p > m$, SINR_k in (2) is still valid. The same SVD yields

$$\mathrm{SINR}_k = \frac{1}{m}\sum_{i=1}^{m} \frac{\|t_{k,i}\|^2}{1+\frac{1}{m}d_i^2} = T_k, \quad (13)$$

because $\mathbf{H}_{(-k)}$ has only m nonzero singular values when $p > m$. The results from random matrix theory that we will use later also apply to $p > m$. However, in this study, we restrict our attention to $p \le m$.

III. Distributions of S_k and t_k

The distribution of S_k and its relationship with T_k can be obtained from the properties of the multivariate Normal distribution (see [18] Chapter 3). Here, $\mathbf{R} = \mathbf{P}^{\frac{1}{2}}\mathbf{R}_t\mathbf{P}^{\frac{1}{2}}$, which is assumed to be positive definite, can be considered as the generalized covariance matrix.

We denote a Normal distribution as $\mathrm{N}(\mathrm{mean}, \mathrm{Var})$, and a complex Normal distribution as $\mathrm{CN}(\mathrm{mean}, \mathrm{Var})$. Thus, each row of $\mathbf{H_W} \sim \mathrm{CN}(\mathbf{0}, \mathbf{I}_p)$, and each row of $\mathbf{H} \sim \mathrm{CN}(\mathbf{0}, \mathbf{R})$.

Observe that,

$$\begin{aligned}
\Sigma_k &\triangleq \frac{1}{[\mathbf{R}^{-1}]_{kk}} = \frac{\tilde{c}_k}{[\mathbf{R_t}^{-1}]_{kk}} \\
&= r_{kk} - r_{k(-k)}^\dagger (\mathbf{R}_{(-k,-k)})^{-1} r_{k(-k)}, \quad (14)
\end{aligned}$$

where r_{kk} is the $(k,k)^{th}$ element of $\mathbf{R}$, $r_{k(-k)}$ is the k^{th} column of $\mathbf{R}$ with the k^{th} element removed, $\mathbf{R}_{(-k,-k)}$ is $\mathbf{R}$ with the k^{th} row and k^{th} column removed, $[\mathbf{R}^{-1}]_{kk}$ is the $(k,k)^{th}$ entry of $\mathbf{R}^{-1}$.

Conditional on $\mathbf{H}_{(-k)}$, h_k, t_k, and s_k are all Normally distributed:

$$\begin{aligned}
h_k|\mathbf{H}_{(-k)} &\sim -\mathbf{H}_{(-k)}(\mathbf{R}_{(-k,-k)})^{-1} r_{k(-k)} \\
&\quad + \mathrm{CN}\left(\mathbf{0}, \Sigma_k \mathbf{I}_m\right), \quad (15) \\
t_k = \mathbf{U}^\dagger h_k|\mathbf{H}_{(-k)} &\sim -\mathbf{D}\mathbf{V}^\dagger(\mathbf{R}_{(-k,-k)})^{-1} r_{k(-k)} \\
&\quad + \mathrm{CN}\left(\mathbf{0}, \Sigma_k \mathbf{I}_{p-1}\right), \quad (16) \\
s_k = \mathbf{U}_c^\dagger h_k|\mathbf{H}_{(-k)} &\sim \ \mathrm{CN}\left(\mathbf{0}, \Sigma_k \mathbf{I}_{m-p+1}\right). \quad (17)
\end{aligned}$$

(Recall that $\mathbf{H}_{(-k)} = \mathbf{U}\mathbf{D}\mathbf{V}^\dagger$, $\mathbf{U}^\dagger\mathbf{H}_{(-k)} = \mathbf{D}\mathbf{V}^\dagger$, $\mathbf{U}_c^\dagger\mathbf{H}_{(-k)} = \mathbf{U}_c^\dagger\mathbf{U}\mathbf{D}\mathbf{V}^\dagger = \mathbf{0}$.)

We state our first lemma.

Lemma 1: $\mathrm{SINR}_k^{\mathrm{ZF}} = S_k$ is a Gamma random variable, $S_k \sim \mathrm{G}(m-p+1, \frac{1}{m}\Sigma_k)$. S_k is independent of T_k.

Proof: Recall that $S_k = \frac{1}{m}\sum_{i=1}^{m-p+1}\|s_{k,i}\|^2$. The independence of S_k and T_k follows from (16) and (17). Conditional distribution of s_k given $\mathbf{H}_{(-k)}$ is independent of $\mathbf{H}_{(-k)}$. Therefore, (17) is the unconditional distribution of s_k. The elements of s_k are i.i.d. $\mathrm{CN}(\mathbf{0}, \Sigma_k)$, and therefore $S_k \sim \mathrm{G}(m-p+1, \frac{1}{m}\Sigma_k)$.

■

Gore *et. al.* [19] proved that $\mathrm{SINR}_k^{\mathrm{ZF}}$ is a Gamma random variable for the equal power case. Both Müller *et. al.* [17] and Tse and Zeitouni [3] used a Beta distribution to approximate $\mathrm{SINR}_k^{\mathrm{ZF}}$.

The first three moments of S_k are

$$\begin{aligned}
\mathrm{E}(S_k) &= \frac{m-p+1}{m}\Sigma_k, \\
\mathrm{Var}(S_k) &= \frac{m-p+1}{m^2}\Sigma_k^2, \\
\mathrm{Sk}(S_k) &= \mathrm{E}\left((S_k - \mathrm{E}(S_k))^3\right) = 2\frac{m-p+1}{m^3}\Sigma_k^3. \quad (18)
\end{aligned}$$

The following lemma concerns the distribution of t_k.

Lemma 2: Conditional on $\mathbf{H}_{(-k)}$, $\|t_{k,i}\|^2$ is a non-central Chi-squared random variable with a moment generating function (MGF)

$$\mathrm{E}\left(\exp(\|t_{k,i}\|^2 y)\middle|\, \mathbf{H}_{(-k)}\right) = \frac{1}{1-y\Sigma_k}\exp\left(\frac{y\|z_i\|^2}{1-y\Sigma_k}\right). \quad (19)$$

where $z_i = -d_i\left(\mathbf{V}^\dagger\mathbf{R}_{(-k,-k)}^{-1} r_{k(-k)}\right)_i$. For uncorrelated channels, $\|t_{k,i}\|^2 \sim \mathrm{G}\left(1, \tilde{c}_k\right)$, independent of $\mathbf{H}_{(-k)}$.

Proof: From (16), $t_{k,i}|\mathbf{H}_{(-k)} \sim \mathrm{CN}(z_i, \Sigma_k)$. For uncorrelated channels, $z_i = \mathbf{0}$ and $\Sigma_k = \tilde{c}_k$ hence $\|t_{k,i}\|^2 \sim \mathrm{G}(1, \tilde{c}_k)$. In general, conditional on $\mathbf{H}_{(-k)}$, $\frac{2}{\Sigma_k}\|t_{k,i}\|^2$ is a standard non-central Chi-squared random variable with 2 degrees of freedom and non-centrality parameter $\frac{2}{\Sigma_k}\|z_i\|^2$, which gives the MGF in (19). ∎

For future use we give expressions for three moments of $\|t_{k,i}\|^2$, conditional on $\mathbf{H}_{(-k)}$.

$$\begin{aligned}\mathrm{E}(\|t_{k,i}\|^2|\mathbf{H}_{(-k)}) &= \Sigma_k + \|z_i\|^2,\\ \mathrm{Var}(\|t_{k,i}\|^2|\mathbf{H}_{(-k)}) &= \Sigma_k^2 + 2\|z_i\|^2\Sigma_k,\\ \mathrm{Sk}(\|t_{k,i}\|^2|\mathbf{H}_{(-k)}) &= 2\Sigma_k^3 + 6\|z_i\|^2\Sigma_k^2. \end{aligned} \tag{20}$$

The significance of the decomposition of SINR_k can be seen from Fig. 1, which plots $\mathrm{E}\left(\frac{T}{S}\right)$ as a function of p, c (SNR), and correlation ρ, for $m = 16$. Here we consider equal power $c_k = c$ case with $\mathbf{R}_t$ being an equicorrelation matrix (i.e., $\mathbf{R}_t$ consists of 1's in the main diagonal and ρ's in all off diagonal entries). This figure suggests that in the major range of SNR and $\frac{p}{m}$, S_k might be the dominating component.

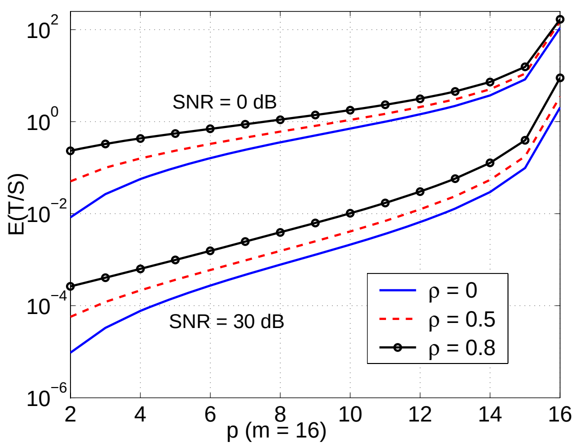


Fig. 1. The ratios $\mathrm{E}\left(\frac{T}{S}\right)$, obtained from 10^5 simulations, with respect to $\frac{p}{m}$, for two (equal power) SNR levels ($c = 0$ dB, 30 dB), and three levels of equicorrelations ($\rho = 0,\ 0.5,\ 0.8$).

IV. Asymptotic Moments

This section derives the asymptotic moments of T_k, written as

$$T_k = \frac{1}{m}\sum_{i=1}^{p-1}\frac{\|t_{k,i}\|^2}{1+\frac{1}{m}d_i^2} = \frac{1}{m}\sum_{i=1}^{p-1}\|t_{k,i}\|^2\lambda_i = \frac{1}{m}t_k^\dagger\mathbf{\Lambda}t_k, \tag{21}$$

where $\lambda_i = \frac{1}{1+\frac{1}{m}d_i^2}$, $\mathbf{\Lambda} = \left(\mathbf{I}_{p-1} + \frac{1}{m}\mathbf{D}\right)^{-1} = \mathrm{diag}[\lambda_1, ..., \lambda_{p-1}]$.

We use some known results for the empirical eigenvalue distribution (ESD) of the product of two random matrices (e.g, Silverstein [20], Bai [21], Silverstein and Bai [22]), to study the asymptotic properties of T_k. We work under the regime: $p \to \infty, m \to \infty, \frac{p-1}{m} \to \gamma \in (0,1)$. In the rest of the paper, whenever we mention "in the limit", we refer to this condition.

Suppose that $\{\tau_i\}_{i=1}^{p-1}$ are the eigenvalues of $\mathbf{R}_{(-k,-k)}$. Suppose further that the ESD of $\mathbf{R} = \mathbf{P}^{\frac{1}{2}}\mathbf{R_t}\mathbf{P}^{\frac{1}{2}}$ converges as $p \to \infty$ to a non-random measure F^R. By Theorem 1.1 of [20], under some weak regularity conditions on F^R (including that the support of F^R is compact and does not contain 0), in the limit, the ESD of $\frac{1}{m}\mathbf{H}_{(-k)}^\dagger\mathbf{H}_{(-k)}$, denoted by $\hat{J}$, converges to a measure J, whose Stieltjes transform, denoted by M_J, satisfies

$$M_J(z) \triangleq \int \frac{1}{x-z}J(dx) = \int \frac{dF^R(\tau)}{\tau(1-\gamma-\gamma z M_J(z)) - z}, \tag{22}$$

(see [8]). For finite p, the last integral in (22) can be approximated by

$$\frac{1}{p-1}\sum_{i=1}^{p-1}\frac{1}{\tau_i(1-\gamma-\gamma z M_J(z)) - z}.$$

Note that for uncorrelated channels, $\tau_i = \tilde{c}_i$. In general, (22) requires that $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, and $M_J(z)$ is the unique solution in $\{M \in \mathbb{C} : -\frac{1-\gamma}{z} + \gamma M \in \mathbb{C}^+\}$. However, since $M_J(-1) \geq 0$ and since $\mathbf{R}$ is positive definite (implying that all τ_i's are positive), and we only consider $\gamma \leq 1$, it follows that $\frac{1}{\tau_i(1-\gamma-\gamma z M_J(z))-z}$ and its derivatives are bounded in a complex neighborhood of $z = -1$, hence $M_J(z)$ and its derivatives are well-defined at $z = -1$ by the bounded convergence theorem.

Therefore,

$$\begin{aligned}\frac{\mathrm{tr}(\mathbf{\Lambda})}{p-1} &= \frac{1}{p-1}\sum_{i=1}^{p-1}\frac{1}{1+\frac{1}{m}d_i^2} = \int\frac{1}{1+x}\hat{J}(dx)\\ &= \int\frac{1}{x-(-1)}\hat{J}(dx) \xrightarrow{p} M_J(-1) \triangleq \mu_{\gamma,c}, \end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\mathrm{tr}(\mathbf{\Lambda}^2)}{p-1} &= \frac{1}{p-1}\sum_{i=1}^{p-1}\frac{1}{\left(1+\frac{1}{m}d_i^2\right)^2}\\ &= \int\frac{1}{(x-(-1))^2}\hat{J}(dx) \xrightarrow{p} M_J'(-1) \triangleq \sigma_{\gamma,c}^2, \end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\mathrm{tr}(\mathbf{\Lambda}^3)}{p-1} &= \frac{1}{p-1}\sum_{i=1}^{p-1}\frac{1}{\left(1+\frac{1}{m}d_i^2\right)^3}\\ &= \int\frac{1}{(x-(-1))^3}\hat{J}(dx) \xrightarrow{p} \frac{1}{2}M_J''(-1) \triangleq \eta_{\gamma,c}, \end{aligned} \tag{25}$$

where "$\xrightarrow{p}$" denotes "converge in probability", and $M_J'(z)$ and $M_J''(z)$ are the first and second derivatives of $M_J(z)$, respectively. In general, $M_J(z)$, $M_J'(z)$, and $M_J''(z)$ have to be solved numerically except in some simple cases (e.g., an uncorrelated channel with equal powers).

Let $\Delta_i(z) = \tau_i(1-\gamma-\gamma z M_J(z)) - z$. $M_J'(z)$, and M_J'' can be approximated by solving

$$M_J'(z)\left(1-\frac{1}{p-1}\sum_{i=1}^{p-1}\frac{\tau_i\gamma z}{\Delta_i(z)^2}\right) = \frac{1}{p-1}\sum_{i=1}^{p-1}\frac{\tau_i\gamma M_J(z)+1}{\Delta_i(z)^2},$$

$$\begin{aligned}M_J''(z)\left(1-\frac{1}{p-1}\sum_{i=1}^{p-1}\frac{\tau_i\gamma z}{\Delta_i(z)^2}\right) &= \frac{2}{p-1}\sum_{i=1}^{p-1}\frac{\tau_i\gamma M_J'(z)}{\Delta_i(z)^2}\\ &\quad + \frac{2}{p-1}\sum_{i=1}^{p-1}\frac{(\tau_i\gamma M_J(z) + \tau_i\gamma z M_J'(z) + 1)^2}{\Delta_i(z)^3}.\end{aligned}$$

$M_J(z)$ and its derivatives are not sufficient for computing the moments of T_k, which involve $\|t_{k,i}\|^2$. We will combine the conditional moments of $\|t_{k,i}\|^2$ in (20) with $M_J(z)$ to get the asymptotic moments of T_k.

Three cases are considered in the following three subsections. First, we consider uncorrelated channels with unequal powers and obtain the asymptotic moments of T_k explicitly, although the results can not be expressed in closed-forms. Next, we show that, assuming equal powers, the asymptotic moments of T_k for uncorrelated channels have closed-form expressions. Finally, for general correlated channels with unequal powers, we derive some limiting upper bounds for the moments of T_k and give some sufficient conditions under which these upper bounds are the exact limits.

A. Asymptotic Moments of T_k for Uncorrelated Channels

In this case, $\mathbf{R}_t = \mathbf{I}_p$, $\mathbf{R} = \mathbf{P}$, $\Sigma_k = \tilde{c}_k = c_k \frac{m}{p}$, $r_{k(-k)} = 0$, $\|z_i\|^2 = d_i^2 \|(V^\dagger \mathbf{R}_{(-k,-k)}^{-1} r_{k(-k)})_i\|^2 = 0$, and $\|t_{k,i}\|^2 \sim \mathrm{G}(1, \tilde{c}_k)$ as shown in Lemma 2.

The following three lemmas are proved in Appendix I.

Lemma 3:

$$\mathrm{E}\left(\frac{p}{p-1} T_k\right) \to c_k \mu_{\gamma,c} \triangleq c_k M_J(-1). \tag{26}$$

Lemma 4:

$$\mathrm{Var}\left(\frac{p}{\sqrt{p-1}} T_k\right) \to c_k^2 \sigma_{\gamma,c}^2 \triangleq c_k^2 M_J'(-1). \tag{27}$$

Lemma 5:

$$\frac{1}{p-1}\mathrm{Sk}(pT_k) \to 2c_k^3 \eta_{\gamma,c} \triangleq c_k^3 M_J''(-1). \tag{28}$$

Therefore, asymptotically, the first three moments of T_k can be approximated as

$$\mathrm{E}(T_k) \approx c_k \frac{p-1}{p} \mu_{\gamma,c}, \tag{29}$$

$$\mathrm{Var}(T_k) \approx c_k^2 \frac{p-1}{p^2} \sigma_{\gamma,c}^2, \tag{30}$$

$$\mathrm{Sk}(T_k) \approx 2c_k^3 \frac{p-1}{p^3} \eta_{\gamma,c}. \tag{31}$$

Combining with the moments of S_k in (18), and using independence of S_k and T_k, we have

$$\mathrm{E}(\mathrm{SINR}_k) \approx c_k \frac{m-p+1}{p} + c_k \frac{p-1}{p} \mu_{\gamma,c}, \tag{32}$$

$$\mathrm{Var}(\mathrm{SINR}_k) \approx c_k^2 \frac{m-p+1}{p^2} + c_k^2 \frac{p-1}{p^2} \sigma_{\gamma,c}^2, \tag{33}$$

$$\mathrm{Sk}(\mathrm{SINR}_k) \approx 2c_k^3 \frac{m-p+1}{p^3} + 2c_k^3 \frac{p-1}{p^3} \eta_{\gamma,c}. \tag{34}$$

B. Closed-form Asymptotic Moments of T_k for Uncorrelated Channels with Equal Powers

In this case, $\mathbf{H} = \sqrt{\tilde{c}}\mathbf{H_W}$. The ESD of $\frac{1}{m}\mathbf{H}_{(-k)}^\dagger \mathbf{H}_{(-k)}$ converges to the well-known "Marcenko-Pastur Law" (Theorem 2.5 in [21]), from which one can derive closed-form expressions for the moments by (tedious) integration. Alternatively, we directly solve for $M_J(-1)$ from the simplified version (i.e., taking $\tau_i = \tilde{c}$ for all i) of (22) to get

$$\mu_{\gamma,c} = M_J(-1) = \frac{\kappa - (\tilde{c}(1-\gamma)+1)}{2\tilde{c}\gamma}, \tag{35}$$

where $\kappa = \sqrt{\tilde{c}^2(1-\gamma)^2 + 2\tilde{c}(1+\gamma)+1}$.

Similarly,

$$\sigma_{\gamma,c}^2 = M_J'(-1) = \mu_{\gamma,c} - \frac{1+\tilde{c}(1+\gamma)-\kappa}{2\tilde{c}\gamma\kappa}, \tag{36}$$

$$\eta_{\gamma,c} = \frac{1}{2}M_J''(-1) = \sigma_{\gamma,c}^2 - \frac{\tilde{c}}{\kappa^3}. \tag{37}$$

From $1+\tilde{c}(1+\gamma) \geq \kappa$, it follows that

$$\mu_{\gamma,c} \geq \sigma_{\gamma,c}^2 \geq \eta_{\gamma,c}. \tag{38}$$

We always replace γ by $\frac{p-1}{m}$ in our computations. In the literature (e.g., [3], [4], [8]), $\gamma = \frac{p}{m}$ is often used. It can be shown that the choice of γ can have a significant impact for small dimensions. [1]

Our approximate formula (32) for $\mathrm{E}(\mathrm{SINR}_k)$ (denoted as $\tilde{\mu}$) is

$$\begin{aligned}\tilde{\mu} &= c\frac{m-p+1}{p} + c\frac{p-1}{p}\mu_{\gamma,c} \\ &= \tilde{c} - \frac{1}{4}\frac{p-1}{m}\frac{1}{\gamma}\left(\sqrt{\tilde{c}\left(1+\sqrt{\gamma}\right)^2+1} - \sqrt{\tilde{c}\left(1-\sqrt{\gamma}\right)^2+1}\right)^2.\end{aligned} \tag{39}$$

When γ is replaced with $\frac{p-1}{m}$, the corresponding $\tilde{\mu}$ is denoted by $\tilde{\mu}_{p-1}$. If instead γ is replaced with $\frac{p}{m}$, it is denoted by $\tilde{\mu}_p$. We can compare $\tilde{\mu}$ with the well-known asymptotic first moment (e.g. [3], [4] (6.59)), denoted by $\tilde{\mu}_R$,

$$\tilde{\mu}_R = \tilde{c} - \frac{1}{4}\left(\sqrt{\tilde{c}\left(1+\sqrt{\gamma}\right)^2+1} - \sqrt{\tilde{c}\left(1-\sqrt{\gamma}\right)^2+1}\right)^2. \tag{40}$$

Similarly, $\tilde{\mu}_{R,p-1}$ and $\tilde{\mu}_{R,p}$ indicate whether $\gamma = \frac{p-1}{m}$ or $\gamma = \frac{p}{m}$ is used for $\tilde{\mu}_R$. Obviously $\tilde{\mu}_{R,p-1} = \tilde{\mu}_{p-1}$.

Fig. 2 plots $\frac{\tilde{\mu}_{p-1}}{\tilde{\mu}_{R,p}}$ and $\frac{\tilde{\mu}_{p-1}}{\tilde{\mu}_p}$ as functions of m and $\frac{p}{m}$. It is clear that the difference between $\tilde{\mu}_{p-1}$ and $\tilde{\mu}_{R,p}$ at small dimensions can be substantial. For example, when $m=4$, $p=4$, $\frac{\tilde{\mu}_{p-1}}{\tilde{\mu}_{R,p}}$ is almost 3. On the other hand, the difference between $\tilde{\mu}_{p-1}$ and $\tilde{\mu}_p$ is not significant. This experiment implies that our proposed approximate formulae are not only accurate (see simulation results in Section VIII) but also not as sensitive to the choice of γ.

The following Lemma compares $\tilde{\mu}$ with $\tilde{\mu}_R$ algebraically:

Lemma 6: $\tilde{\mu}_p \geq \tilde{\mu}_{p-1} = \tilde{\mu}_{R,p-1} \geq \tilde{\mu}_{R,p}$.

Proof: See Appendix II. ■

Similar results can be obtained if we compare our approximate variance formula (33) with the asymptotic variance given by [3]. See the technical report [23] for details.

C. Asymptotic Upper Bounds for Correlated Channels

Recall that $\mathrm{E}\left(\|t_{k,i}\|^2 \middle| \mathbf{H}_{(-k)}\right) = \Sigma_k + \|z_i\|^2$, where $\|z_i\|^2 = d_i^2 \left\|\left(V^\dagger (\mathbf{R}_{(-k,-k)})^{-1} r_{k(-k)}\right)_i\right\|^2$. There is an im-

[1] Note that in our notations, $\tilde{c} = c\frac{m}{p}$ is equivalent to "$\frac{A^2}{\sigma^2}$" in [4] (6.59) and equivalent to "$\frac{P}{\sigma^2}$" in [3].

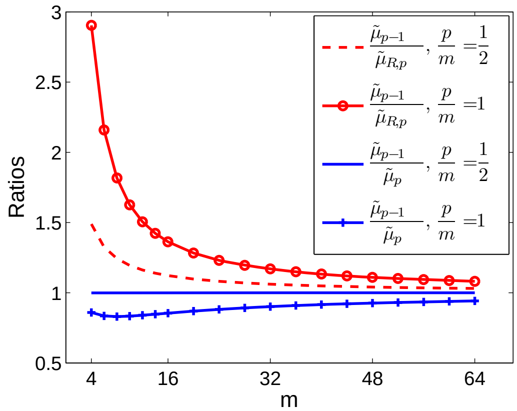


Fig. 2. The ratios $\frac{\tilde{\mu}_{p-1}}{\tilde{\mu}_{R,p}}$ and $\frac{\tilde{\mu}_{p-1}}{\tilde{\mu}_{p}}$, with respect to m and p/m, for SNR = 20 dB.

portant inequality:

$$\begin{aligned}\sum_{i=1}^{p-1}\left(\frac{1}{m}\|z_i\|^2\lambda_i\right) &= \sum_{i=1}^{p-1}\frac{\frac{1}{m}\|z_i\|^2}{1+\frac{1}{m}d_i^2}\\ &\le \sum_{i=1}^{p-1}\left\|\left(V^\dagger(\mathbf{R}_{(-k,-k)})^{-1}r_{k(-k)}\right)_i\right\|^2\\ &= \|V^\dagger\mathbf{R}_{(-k,-k)})^{-1}r_{k(-k)}\|^2\\ &= \|\mathbf{R}_{(-k,-k)})^{-1}r_{k(-k)}\|^2 \triangleq e_k. \end{aligned} \tag{41}$$

Some limiting upper bounds for the first three moments of T_k are provided in the next lemma.

Lemma 7:

$$\mathrm{E}(T_k)^U = \frac{p-1}{m}\Sigma_k M_J(-1) + e_k, \tag{42}$$

$$\mathrm{Var}\,(T_k)^U = \frac{p-1}{m^2}\Sigma_k^2 M_J'(-1) + \frac{1}{m}\Sigma_k\left(\frac{1}{2}+2\sqrt{\frac{2}{\delta}}\right)e_k + e_k^2, \tag{43}$$

$$\begin{aligned}\mathrm{Sk}(T_k)^U &= \frac{p-1}{m^3}\Sigma_k^3 M_J''(-1) + \frac{8/9}{m^2}\Sigma_k^2 e_k + \frac{6}{m^2}\Sigma_k^2\frac{1}{\delta}e_k + e_k^3\\ &\quad + \frac{3}{2}e_k\Sigma_k\frac{1}{m}\sqrt{e_k + 2\frac{1}{m}\Sigma_k\sqrt{\frac{2}{\delta}}}\sqrt{e_k + 2\frac{1}{m}\Sigma_k\sqrt{\frac{2}{\delta'}}}.\end{aligned} \tag{44}$$

where δ and δ' are the same constants in the proofs of Lemma 4 and Lemma 5.

Proof: See Appendix III. ∎

We are interested in the special case where $e_k \to 0$. If $e_k \to 0$ at any rate (faster than $O(p^{-\frac{1}{2}})$ or faster than $O(p^{-\frac{2}{3}})$), ignoring all the terms involving e_k, $\mathrm{E}(T_k)^U$ ($\mathrm{Var}(T_k)^U$ or $\mathrm{Sk}(T_k)^U$) is still the true limit. Under these conditions, we propose the approximate moments for SINR_k

$$\mathrm{E}(\mathrm{SINR}_k) \approx \frac{m-p+1}{m}\Sigma_k + \frac{p-1}{m}\Sigma_k\mu_{\gamma,c} + e_k, \tag{45}$$

$$\mathrm{Var}(\mathrm{SINR}_k) \approx \frac{m-p+1}{m^2}\Sigma_k^2 + \frac{p-1}{m^2}\Sigma_k^2\sigma_{\gamma,c}^2 + e_k^2, \tag{46}$$

$$\mathrm{Sk}(\mathrm{SINR}_k) \approx 2\frac{m-p+1}{m^3}\Sigma_k^3 + 2\frac{p-1}{m^3}\Sigma_k^3\eta_{\gamma,c} + e_k^3, \tag{47}$$

Note that e_k, e_k^2, e_k^3 are retained in these expressions because simulation studies show that keeping these terms improves accuracy at very small dimensions.

It turns out that in the equicorrelation situation, $e_k \to 0$ at a good rate under mild regularity conditions. The equicorrelation is the simplest correlation model (e.g., [15] (4.41)) and is often used to model closely spaced antennas or for the worst case analysis [24]. For more realistic correlation models, see Narasimhan [25].

Lemma 8: If the correlation matrix $\mathbf{R}_t$ consists of 1's in the main diagonal and ρ's in all off-diagonal entries,[2] then $e_k \to 0$ in the limit, as long as $\sum_{i\neq k}\frac{c_k}{c_i} = O(p^f)$ for $f < 2$. Sufficient conditions for $e_k \to 0$ faster than $O(p^{-\frac{1}{2}})$ and $O(p^{-\frac{2}{3}})$ would be $f < \frac{3}{2}$ and $f < \frac{4}{3}$, respectively.

Proof: $\mathbf{R}_t$ can be written as $\mathbf{R}_t = (1-\rho)\mathbf{I}_p + \rho\mathbf{1}_p\mathbf{1}_p^T$, where $\mathbf{1}_p$ denotes a column vector of p 1's. Recall that $\mathbf{R} = \mathbf{P}^{\frac{1}{2}}\mathbf{R}_t\mathbf{P}^{\frac{1}{2}}$. $\mathbf{R}_{(-k,-k)} = (1-\rho)\mathbf{P}_{(-k,-k)} + \rho\mathbf{P}^{\frac{1}{2}}_{(-k,-k)}\mathbf{1}_{p-1}\mathbf{1}_{p-1}^T\mathbf{P}^{\frac{1}{2}}_{(-k,-k)}$, with an inverse

$$\begin{aligned}&(\mathbf{R}_{(-k,-k)})^{-1}\\ &= \frac{\mathbf{P}^{-1}_{(-k,-k)}}{1-\rho} - \frac{\mathbf{P}^{-\frac{1}{2}}_{(-k,-k)}}{1-\rho}\frac{\rho}{(p-1)\rho+1-\rho}\mathbf{1}_{p-1}\mathbf{1}_{p-1}^T\mathbf{P}^{-\frac{1}{2}}_{(-k,-k)}\end{aligned} \tag{48}$$

Note that $r_{k(-k)} = \rho\mathbf{P}^{\frac{1}{2}}_{kk}\mathbf{P}^{\frac{1}{2}}_{(-k,-k)}\mathbf{1}_{p-1}$. Therefore

$$\begin{aligned}&\left(\mathbf{R}_{(-k,-k)}\right)^{-1}r_{k(-k)}\\ &= \frac{\rho}{1-\rho}\mathbf{P}^{\frac{1}{2}}_{kk}\mathbf{P}^{-\frac{1}{2}}_{(-k,-k)}\mathbf{1}_{p-1}\\ &\quad - \frac{\rho}{1-\rho}\mathbf{P}^{\frac{1}{2}}_{kk}\mathbf{P}^{-\frac{1}{2}}_{(-k,-k)}\frac{\rho}{(p-1)\rho+1-\rho}\mathbf{1}_{p-1}(p-1)\\ &= \frac{\rho}{(p-1)\rho+1-\rho}\mathbf{P}^{\frac{1}{2}}_{kk}\mathbf{P}^{-\frac{1}{2}}_{(-k,-k)}\mathbf{1}_{p-1},\end{aligned} \tag{49}$$

and

$$\begin{aligned}e_k &= \|(\mathbf{R}_{(-k,-k)})^{-1}r_{k(-k)}\|^2\\ &= \left(\frac{\rho}{(p-1)\rho+1-\rho}\right)^2\mathbf{P}_{kk}\mathbf{1}_{p-1}^T\mathbf{P}^{-1}_{(-k,-k)}\mathbf{1}_{p-1}\\ &= \left(\frac{\rho}{(p-1)\rho+1-\rho}\right)^2\sum_{i\neq k}\frac{c_k}{c_i},\end{aligned} \tag{50}$$

from which it follows that $e_k \to 0$ if $\sum_{i\neq k}\frac{c_k}{c_i} = O(p^f)$, for some $f < 2$. If we want $e_k \to 0$ faster than $O(p^{-\frac{1}{2}})$ and $O(p^{-\frac{2}{3}})$, we should have $f < \frac{3}{2}$ and $f < \frac{4}{3}$, respectively. This completes the proof. ∎

In the equal power case, $\sum_{i\neq k}\frac{c_k}{c_i} = p-1$, and $\Sigma_k \approx (1-\rho)\tilde{c}$, which implies that the first (second, third) moment of SINR_k for an equicorrelation channel is roughly only $1-\rho$ ($(1-\rho)^2$, $(1-\rho)^3$) of the moment without correlations.

To conclude this section, we mention that our asymptotic results also apply to the real channels (e.g., in [3], [8]) with a

[2] In order for $\mathbf{R}_t$ to be positive definite, we have to restrict $-\frac{1}{p-1} < \rho < 1$.

modification that involves multiplying the variance by a factor of 2 and the third moment by a factor of 4.

V. Asymptotic Normality of T_k

This section proves that, if appropriately normalized, T_k converges in distribution to a Normal random variable, both in the case of uncorrelated channels as well as the correlated channels if $e_k \to 0$ faster than $O(p^{-\frac{1}{2}})$. The latter includes the equicorrelation channel with regularity conditions as in Lemma 8.

The difficulty in dealing with channels with arbitrary correlations is due to the term

$$\begin{aligned}&\sum_{i=1}^{p-1}\frac{1}{m}\|z_i\|^2\lambda_i\\ =&\sum_{i=1}^{p-1}\left(\frac{\frac{1}{m}d_i^2}{1+\frac{1}{m}d_i^2}\right)\left\|\left(\mathbf{V}^\dagger(\mathbf{R}_{(-k,-k)})^{-1}r_{k(-k)}\right)_i\right\|^2. \qquad (51)\end{aligned}$$

Although (51) is bounded by $e_k = \|(R_{(-k,-k)})^{-1}r_{k(-k)}\|^2$, it is challenging to show that (51) converges (not necessarily to zero). For the rest of this section, we assume that $e_k \to 0$ faster than $O(p^{-\frac{1}{2}})$, which makes the contribution of the term (51) asymptotically negligible.

Lemma 9:

$$W = \frac{mT_k - \Sigma_k \text{tr}(\mathbf{\Lambda})}{\sqrt{p-1}} \overset{D}{\Longrightarrow} \text{N}\left(0, \Sigma_k^2\sigma_{\gamma,c}^2\right) \qquad (52)$$

where "$\overset{D}{\Longrightarrow}$" means "converge in distribution".

Proof: See Appendix IV. ■

Corollary 1:

$$\frac{mT_k - (p-1)\Sigma_k\mu_{\gamma,c}}{\sqrt{p-1}} \overset{D}{\Longrightarrow} \text{N}\left(0, \Sigma_k^2\sigma_{\gamma,c}^2\right) \qquad (53)$$

Proof:

$$\begin{aligned}&\frac{mT_k - (p-1)\Sigma_k\mu_{\gamma,c}}{\sqrt{p-1}}\\ =&\frac{mT_k - \Sigma_k\text{tr}\left(\mathbf{\Lambda}\right)}{\sqrt{p-1}} + \frac{\Sigma_k\text{tr}\left(\mathbf{\Lambda}\right) - (p-1)\Sigma_k\mu_{\gamma,c}}{\sqrt{p-1}}.\end{aligned}$$

We have shown that $\frac{mT_k - \Sigma_k\text{tr}(\mathbf{\Lambda}_c)}{\sqrt{p-1}} \overset{D}{\Longrightarrow} \text{N}\left(0, \Sigma_k^2\sigma_{\gamma,c}^2\right)$. The fact that $\Sigma_k\frac{\text{tr}(\mathbf{\Lambda})-(p-1)\mu_{\gamma,c}}{\sqrt{p-1}} \overset{P}{\to} 0$, follows from Theorem 1.1 in Bai and Silverstein [22], which implies that $(p-1)\left(\frac{\text{tr}(\mathbf{\Lambda})}{p-1} - \mu_{\gamma,c}\right)$ is stochastically bounded. Equation (53) now follows from the Converging Together Lemma (or Slutsky's Theorem) [26]. ■

Corollary 2:

$$\frac{\text{SINR}_k - \frac{m-p+1}{m}\Sigma_k - \frac{p-1}{m}\Sigma_k\mu_{\gamma,c}}{\sqrt{\frac{m-p+1}{m^2}\Sigma_k^2 + \frac{p-1}{m^2}\Sigma_k^2\sigma_{\gamma,c}^2}} \overset{D}{\Longrightarrow} \text{N}(0,1). \qquad (54)$$

Proof: This is a direct consequence of the independence of S_k and T_k. ■

VI. Distribution Approximations

Once the asymptotic Normality of T_k (and SINR_k) is proved, it is possible to rigorously define the "approximating" distributions. Since S_k and T_k are independent, we do not need to approximate the distribution of S_k. However, we keep the option of approximating the distribution of $S_k + T_k$ itself.

Our approach is to match the first two asymptotic moments of T_k with the corresponding moments of a target distribution (e.g., Gamma). The asymptotic Normality of T_k (Lemma 9) ensures that the approximate distribution converges to the true limit. The Normal approximation is not accurate for small dimensions, and to an extent this is because T_k is positive and a Normal random variable has zero third central moment. We expect a Gamma distribution, which has non-negative support and non-zero third central moment, to be a better approximation.

It turns out that only considering the first two moments may not be sufficient for accurately computing error probabilities. Therefore, we consider a generalized Gamma distribution, which matches the first three moments and hence is likely produce better results, as another approximation.

In this section, the same conditions are assumed as in proving the asymptotic Normality of T_k. That is, $e_k \to 0$ faster than $O(p^{-\frac{1}{2}})$. When this condition is not satisfied, we do not have a rigorous asymptotic result.

In this section, the symbol $\sim$ is also used to denote the approximate distributions.

A. Normal Approximation

Asymptotic Normality of T_k implies that,

$$T_k \sim \text{N}\left(\frac{p-1}{m}\Sigma_k\mu_{\gamma,c},\ \frac{p-1}{m^2}\Sigma_k^2\sigma_{\gamma,c}^2\right). \qquad (55)$$

Also, the asymptotic Normality of SINR_k means that,

$$\begin{aligned}\text{SINR}_k \sim \text{N}\Bigg(&\frac{m-p+1}{m}\Sigma_k + \frac{p-1}{m}\Sigma_k\mu_{\gamma,c}\ ,\\ &\frac{m-p+1}{m^2}\Sigma_k^2 + \frac{p-1}{m^2}\Sigma_k^2\sigma_{\gamma,c}^2\Bigg). \qquad (56)\end{aligned}$$

B. Gamma Approximation

T_k can be approximated by a Gamma random variable $\text{G}(\alpha_{T_k}, \beta_{T_k})$ whose parameters are determined by solving:

$$\begin{aligned}\text{E}(T_k) = \alpha_{T_k}\beta_{T_k} &= \frac{p-1}{m}\Sigma_k\mu_{\gamma,c},\\ \text{Var}(T_k) = \alpha_{T_k}\beta_{T_k}^2 &= \frac{p-1}{m^2}\Sigma_k^2\sigma_{\gamma,c}^2.\end{aligned}$$

The Gamma approximation of T_k is therefore,

$$T_k \sim \text{G}\left(\frac{(p-1)\mu_{\gamma,c}^2}{\sigma_{\gamma,c}^2},\ \frac{1}{m}\Sigma_k\frac{\sigma_{\gamma,c}^2}{\mu_{\gamma,c}}\right). \qquad (57)$$

According to the Gamma approximation, the third central moment of T_k should be

$$2\alpha_{T_k}\beta_{T_k}^3 = 2\frac{p-1}{m^3}\Sigma_k^3\frac{\sigma_{\gamma,c}^4}{\mu_{\gamma,c}}. \qquad (58)$$

We can also approximate SINR_k by a Gamma distribution $\text{G}(\alpha, \beta)$, again by matching the first two moments,

$$\text{SINR}_k \sim \text{G}\left(\frac{(m-p+1+(p-1)\mu_{\gamma,c})^2}{m-p+1+(p-1)\sigma^2_{\gamma,c}},\right.$$
$$\left.\frac{1}{m}\Sigma_k\frac{m-p+1+(p-1)\sigma^2_{\gamma,c}}{m-p+1+(p-1)\mu_{\gamma,c}}\right). \quad (59)$$

The third central moment of SINR_k then would be

$$\frac{2}{m^3}\Sigma_k^3\frac{(m-p+1+(p-1)\sigma^2_{\gamma,c})^2}{m-p+1+(p-1)\mu_{\gamma,c}}. \quad (60)$$

Define RS to be the ratio of the third central moment of the approximated Gamma distribution to the asymptotic third central moment. For T_k this becomes,

$$RS_{T_k} = \frac{\sigma^4_{\gamma,c}}{\mu_{\gamma,c}\eta_{\gamma,c}}. \quad (61)$$

For SINR_k,

$$RS_{\text{SINR}_k} = \frac{\left(1-\frac{p-1}{m}+\frac{p-1}{m}\sigma^2_{\gamma,c}\right)^2}{\left(1-\frac{p-1}{m}+\frac{p-1}{m}\mu_{\gamma,c}\right)\left(1-\frac{p-1}{m}+\frac{p-1}{m}\eta_{\gamma,c}\right)}. \quad (62)$$

RS_{T_k} and RS_{SINR_k} are indicators of how well the Gamma approximation captures the skewness of the distribution of T_k. Ideally, we would like $RS = 1$. It will be shown later that RS is also critical for the generalized Gamma approximation.

The following inequalities hold for uncorrelated channels with equal powers.

Lemma 10: Assuming equal powers and no correlations,

$$RS_{T_k} \leq 1 \quad (63)$$
$$RS_{\text{SINR}_k} \leq 1 \quad \text{for any } p \leq m \quad (64)$$
$$RS_{T_k} \leq RS_{\text{SINR}_k} \quad \text{in the limit.} \quad (65)$$

Proof: See Appendix V. ∎

Fig. 3 plots some examples of RS_{T_k} and RS_{SINR_k}, for the equal power uncorrelated cases.

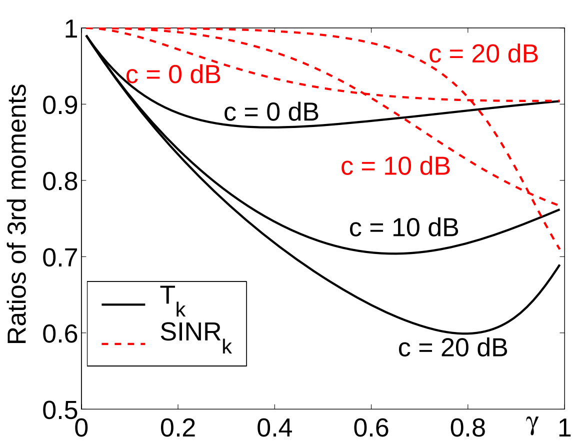


Fig. 3. RS_{T_k} and RS_{SINR_k} for selected SNR levels, over the whole range of γ, for equal power uncorrelated channels.

C. Generalized Gamma Approximation

A generalized Gamma distribution can be described by a stable law or an infinite divisible distribution [27], [28], [26] (Chapter 2.7, 2.8), which involves the sum of i.i.d. sequence of random variables. In our case, conditionally, T_k is a weighted sum of non-i.i.d. random variables, hence T_k is not exactly a stable law nor infinite divisible.

The Gamma approximation of T_k can be generalized by introducing an additional parameter to the original Gamma distribution [27], [28], i.e., assuming $T_k \sim \text{G}(\alpha_{T_k}, \beta_{T_k}, \xi_{T_k})$. The regular Gamma distribution is a special case with $\xi_{T_k} = 1$. Assuming a generalized Gamma distribution, the first three moments of T_k would be

$$\begin{aligned} \text{E}(T_k) &= \alpha_{T_k}\beta_{T_k}, \\ \text{Var}(T_k) &= \alpha_{T_k}\beta^2_{T_k}, \\ \text{Sk}(T_k) &= (\xi_{T_k}+1)\alpha_{T_k}\beta^3_{T_k}. \end{aligned} \quad (66)$$

Equating these moments with the asymptotic moments of T_k will lead to the same α_{T_k}, and β_{T_k} as for the Gamma approximation. The third parameter ξ_{T_k} will be

$$\xi_{T_k} = \frac{2}{RS_{T_k}} - 1. \quad (67)$$

Similarly, we can also generalize the Gamma approximation of SINR_k by assuming $\text{SINR}_k \sim \text{G}(\alpha, \beta, \xi)$. The third parameter will be

$$\xi = \frac{2}{RS_{\text{SINR}_k}} - 1. \quad (68)$$

When $\xi > 1$, the generalized Gamma distribution with these parameter does not have an explicit density in general. However, it can be described in terms of the stable law and has a closed-form MGF [27], [28],

$$\begin{aligned} &\text{MGF}(s; \text{G}(\alpha, \beta, \xi)) \\ &= \exp\left(\frac{\alpha}{\xi-1}\left(1-(1-\beta\xi s)^{\frac{\xi-1}{\xi}}\right)\right). \end{aligned} \quad (69)$$

When $\xi < 1$, the generalized Gamma distribution is a compound Poisson distribution with a MGF

$$\begin{aligned} &\text{MGF}(s; \text{G}(\alpha, \beta, \xi)) \\ &= \exp\left(\frac{\alpha}{1-\xi}\left(\left(\frac{1}{1-\beta\xi s}\right)^{\frac{1-\xi}{\xi}}-1\right)\right). \end{aligned} \quad (70)$$

From Lemma 10, it follows that $\xi_{T_k} > 1$ and $\xi > 1$ for uncorrelated equal power channels.

VII. Analysis of the Probability of Error

Computation of the probability of errors using the distribution of SINR_k is a way of measuring how successfully the proposed distribution approximates the truth. For this purpose, we will compute the BER using (6), which is equivalent to the BFSK BER. It should be straightforward to apply our methods to other types of (non-binary) constellations.

To simplify the notations, the subscript k in BER_k will be dropped for the rest of the paper. In this section, we will provide BER formulae under the Gamma and the generalized

Gamma approximations, and compare these results with the exact formula for BER, denoted by BER_e, given in [4] (6.47) or [7].

If the asymptotic Normality of SINR_k holds, as in Corollary 2, then

$$\begin{aligned}\text{BER}_a \triangleq &\int_0^\infty \left(\int_{\sqrt{x}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\right) dF_{\text{SINR}_k}(x) \\ \to &\text{BER}_\infty \triangleq Q\left(\sqrt{\text{E}(\text{SINR}_k)_\infty}\right). \end{aligned} \quad (71)$$

For uncorrelated channels with equal powers,

$$\begin{aligned}&\text{E}(\text{SINR}_k)_\infty \\ =&\frac{\sqrt{c^2(1-\gamma)^2+2c(1+\gamma)+\gamma^2}+c(1-\gamma)-\gamma}{2\gamma}, \end{aligned} \quad (72)$$

by treating $\frac{p-1}{m} = \frac{p}{m} = \gamma$. However, for correlated channels, since it is not convenient to compute $\text{E}(\text{SINR}_k)_\infty$, we will use the our finite dimensional moment formula to compute $\text{E}(\text{SINR}_k)_\infty$, which is different for different p and m.

Next, we will derive a variety of BER formulae corresponding to various approximation schemes, based on BFSK, which can be easily generalized to other types of constellations.

A. BER by Gamma Approximation on SINR$_k$

Denote the BER computed using the Gamma approximation by BER_g. Integration of (6) by parts yields

$$\text{BER}_g = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} F_{\text{SINR}_k}(x) \frac{1}{2} x^{-\frac{1}{2}} dx. \quad (73)$$

Replacing $F_{\text{SINR}_k}(x)$ with $F_{\alpha,\beta}(x)$, the CDF of the Gamma distribution $\text{G}(\alpha,\beta)$ as defined in (59), and using the results from integral tables [29], we have

$$\begin{aligned}\text{BER}_g =&\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} F_{\alpha,\beta}(x) \frac{1}{2} x^{-\frac{1}{2}} dx \\ =&\frac{1}{2\Gamma(\alpha)\sqrt{2\pi}} \int_0^\infty x^{-\frac{1}{2}} e^{-\frac{x}{2}} \Gamma\left(\alpha, \frac{x}{\beta}\right) dx \\ =&\frac{1}{2\Gamma(\alpha)\sqrt{2\pi}} \frac{\Gamma(1/2+\alpha)(1/\beta)^\alpha}{\alpha(1/2+1/\beta)^{1/2+\alpha}} \times \\ &{}_2F_1\left(1, 1/2+\alpha, \alpha+1, \frac{1/\beta}{1/2+1/\beta}\right), \end{aligned} \quad (74)$$

where the gamma function $\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt$, and the incomplete gamma function $\Gamma(\alpha, y) = \int_0^y t^{\alpha-1} e^{-t} dt$, and ${}_2F_1(.)$ is the hypergeometric function,

$$\begin{aligned}&{}_2F_1\left(1, 1/2+\alpha, \alpha+1, \frac{1/\beta}{1/2+1/\beta}\right) \\ =&\sum_{n=0}^\infty \frac{\frac{\Gamma(1/2+\alpha+n)}{\Gamma(1/2+\alpha)}}{\frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)}} \frac{\left(\frac{1/\beta}{1/2+1/\beta}\right)^n}{n!}. \end{aligned} \quad (75)$$

B. BER by Generalized Gamma Approximation on SINR$_k$

According to the results of [30] , [31] (Chapter 9.2.3), under the generalized Gamma approximation on SINR_k, the BFSK BER (denoted as BER_{gg}) can be expressed in terms of the MGF, as

$$\begin{aligned}\text{BER}_{gg} =& \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \text{MGF}\left(-\frac{1}{2\sin^2\phi}; \text{G}(\alpha,\beta,\xi)\right) d\phi \\ =&\frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp\left(\frac{\alpha}{\xi-1}\left(1-\left(1+\frac{\beta\xi}{2\sin^2\phi}\right)^{\frac{\xi-1}{\xi}}\right)\right) d\phi, \end{aligned} \quad (76)$$

which is for $\xi > 1$ and can be evaluated numerically. We can similarly write down BER_{gg} for $\xi < 1$.

C. BER by Generalized Gamma Approximation on T_k

Under this approximation, when $\xi_{T_k} > 1$, we can write down the MGF for $\text{SINR}_k = S_k + T_k$ as

$$\begin{aligned}&\text{MGF}\left(s; \text{G}\left(m-p+1, \frac{\Sigma_k}{m}\right) + \text{G}(\alpha_{T_k}, \beta_{T_k}, \xi_{T_k})\right) \\ =&\frac{1}{\left(1-\frac{\Sigma_k}{m}s\right)^{m-p+1}} \exp\left(\frac{\alpha_{T_k}}{\xi_{T_k}-1}\left(1-(1-\beta_{T_k}\xi_{T_k}s)^{\frac{\xi_{T_k}-1}{\xi_{T_k}}}\right)\right). \end{aligned} \quad (77)$$

The corresponding BER (denoted as BER_{g+gg}) would be

$$\begin{aligned}\text{BER}_{g+gg} =& \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\left(1+\frac{\Sigma_k}{2m\sin^2\phi}\right)^{m-p+1}} \times \\ &\exp\left(\frac{\alpha_{T_k}}{\xi_{T_k}-1}\left(1-\left(1+\frac{\beta_{T_k}\xi_{T_k}}{2\sin^2\phi}\right)^{\frac{\xi_{T_k}-1}{\xi_{T_k}}}\right)\right) d\phi, \end{aligned} \quad (78)$$

which has to be evaluated numerically.

VIII. Simulations

Our simulations consider $m = 4$ ($p = 2, 4$), and $m = 16$ ($p = 8, 16$), for the equal power case. Both uncorrelated and correlated (with equicorrelation $\rho = 0.5$) channels are tested. The range of SNRs is c = 0 dB - 30 dB. Without loss of generality, the first stream (i.e., $k = 1$) is always assumed.

$\mathbf{H_W}$ is sampled 10^6 times for every combination of (m, p, c, and ρ) for computing the empirical moments, distributions, and BER, except when computing the exact BER_e for $m = 16$, $\mathbf{H_W}$ is only sampled 10^5 times.

A. Moments

The theoretical moments are computed using (45), (46) and (47). Fig. 4 plots the first three moments of T_k computed both theoretically and empirically from simulations. For $m = 16$, the theoretical moments match the simulations very well, especially the first moment. When $m = 4$, the curves for the first moment are still quite accurate except for the correlated cases at very small SNRs. Note that due to the $\log$ scale, the errors at small SNRs are largely exaggerated in the figure. When $m = 4$, for the second and third moments, there seem to be some "significant" discrepancies between theoretical results and simulations at large SNRs. However, these errors contribute negligibly to the second and third moments of SINR_k (see Fig. 5). For example, when $m = 4$, $p = 4$,

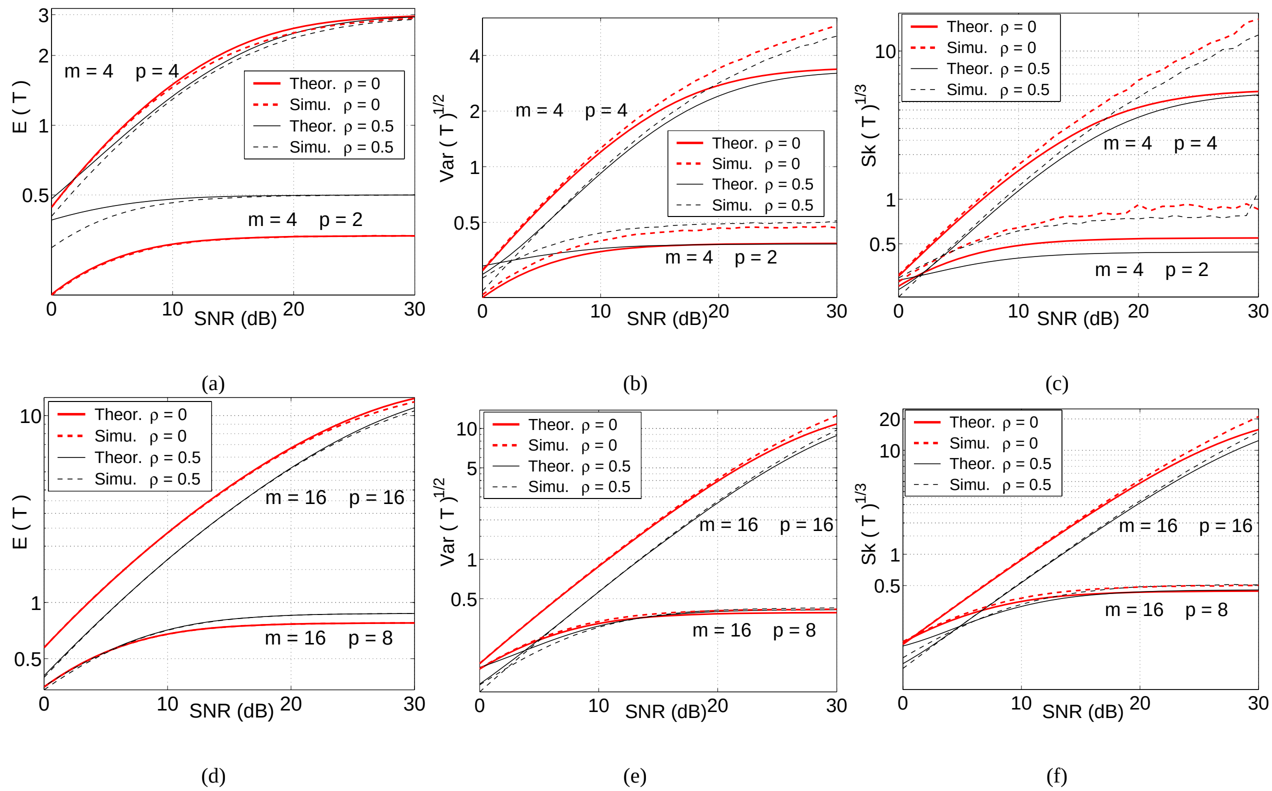


Fig. 4. Theoretical and empirical moments of T_k. The vertical axes are in the $\log_{10}$ scale. Note that the variances and third moments are in terms of their square roots and cubic roots, respectively.

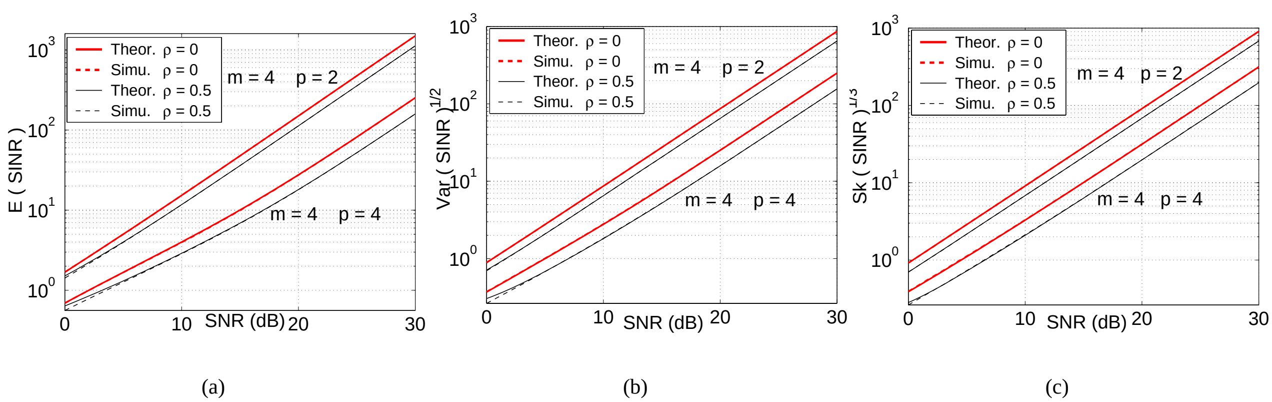


Fig. 5. Theoretical and empirical moments of SINR.

$\rho = 0$, exact $\mathrm{Sk}(S_k) = 314.98^3$. Although the empirical $\mathrm{Sk}(T_k)(= 16.40^3)$ differs quite significantly from the theoretical $\mathrm{Sk}(T_k)(= 5.33^3)$, the theoretical $\mathrm{Sk(SINR)}(= 314.98^3)$ is almost identical to the empirical $\mathrm{Sk(SINR)}(= 314.99^3)$.

The theoretical and empirical moments of SINR_k are compared in Fig. 5 for $m = 4$. As expected, the curves match almost perfectly except for the observable (due to the $\log$ scale) errors at very small SNRs.

B. Distributions

Fig. 6 presents the quantile-quantile (qq) plots for distributions of T_k based on Gamma and Normal approximations against the empirical distribution, at a selected SNR = 10 dB. The figure shows that the Normal approximation works poorly for small m or p. The Gamma approximation fits much better in all cases. Fig. 7 gives the same type of plots for SINR_k. It shows the Gamma approximation works well, especially for $\frac{p}{m} = \frac{1}{2}$. When $\frac{p}{m} = 1$, the Gamma distribution approximates the portion between $1\% - 99\%$ quantiles pretty well.

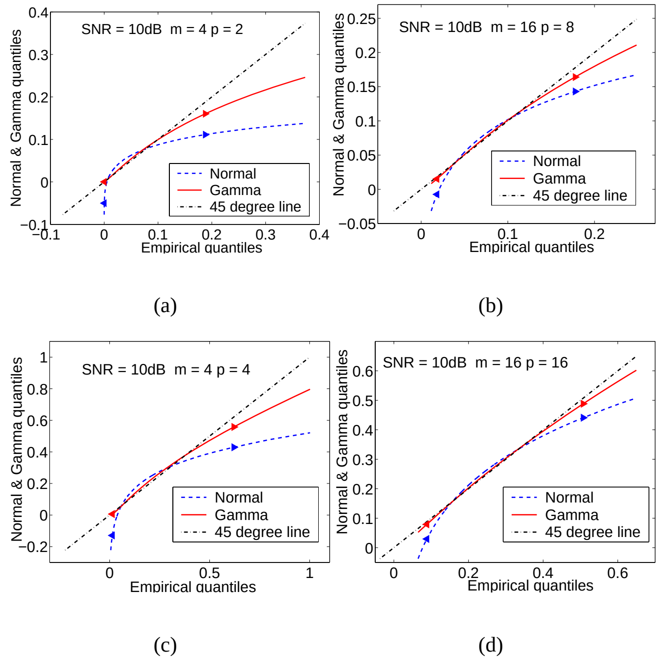


Fig. 6. Quantile-quantile plots for T_k. The triangles on the curves indicate the 1% and 99% quantiles. The range of quantiles is 0.1% - 99.9%.

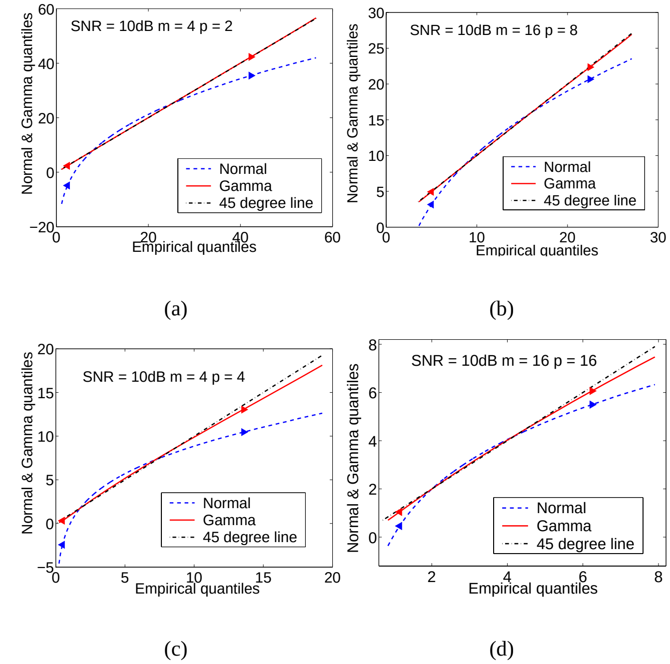


Fig. 7. Quantile-quantile plots for SINR_k. The triangles indicate the 1% and 99% quantiles. The range of quantiles is 0.1% - 99.9%.

We have shown that $RS_{T_k} \leq RS_{\text{SINR}_k}$ for uncorrelated channels with equal powers in Lemma 10, which implies that a Gamma could approximate the distribution of SINR_k better than that of T_k. Also, Fig. 3 shows RS_{SINR_k} is much smaller at $\frac{p}{m} = 1$ than at $\frac{p}{m} = \frac{1}{2}$, which helps explain why Gamma approximation works well at $\frac{p}{m} = \frac{1}{2}$.

C. Error Performance

Fig. 8 plots BERs versus SNRs for uncorrelated channels. The figure shows that the BER curves produced by the Gamma approximation are almost indistinguishable from the simulated curves for $\frac{p}{m} = \frac{1}{2}$. When $\frac{p}{m} = 1$, the Gamma approximation still works well for moderate SNRs (e.g., , SNR < 15 dB). At larger SNRs, the Gamma approximation slightly overestimates BER. The figure also shows that the generalized Gamma distribution produce almost perfect fits to the simulated BER curves. All figures indicate using the asymptotic BER (BER_∞) formula will seriously underestimate the error probabilities at large SNRs. [3]

Fig. 9 presents the BER results for the correlated channels with equicorrelation $\rho = 0.5$. We can see the similar trends as for the uncorrelated cases, i.e., Gamma approximation works well for $\frac{p}{m} = \frac{1}{2}$ and the generalized Gamma approximation performs remarkably well.

Finally, to see the difference between correlated and uncorrelated cases more closely, Fig. 10 plots the interesting portions of the BER curves for both cases, which illustrates that the differences could be significant.

[3]Note that, in order to produce comparable BER results with the classical references (e.g., [2], [4], [8]), we actually used real channels (only for the BER curves).

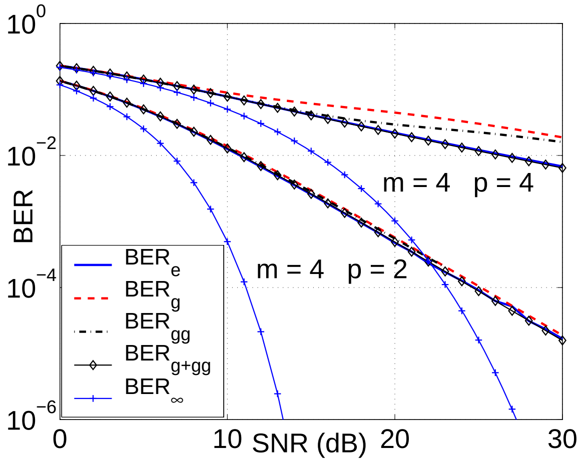
IX. Conclusion

This study characterized the distribution of SINR for the MMSE receiver in the MIMO systems, for channels with non-random transmit correlations and unequal powers. The work started with a key observation that SINR can be decomposed into two independent component: $\text{SINR} = \text{SINR}^{\text{ZF}} + T$, where SINR^{ZF} has an exact Gamma distribution. For uncorrelated channels as well as the correlated channels under certain conditions, T is proved to converge in distribution to a Normal and can be well approximated by a Gamma or a generalized Gamma. Our BER analysis suggested that these approximate distributions can be used to accurately estimate the error probabilities even for very small dimensions.

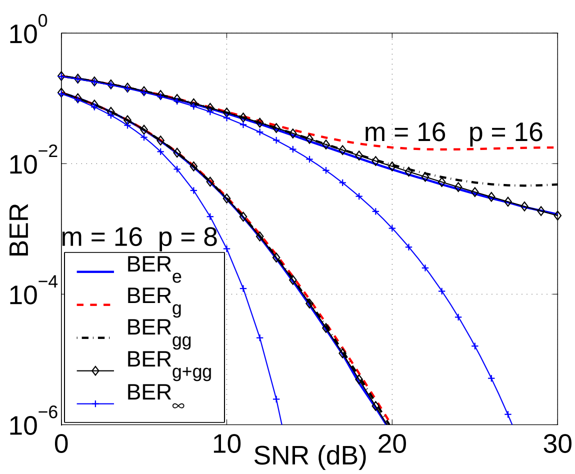
Appendix I
Proof of Lemma 3, 4, 5

Restate *Lemma* 3:

$$\mathrm{E}\left(\frac{p}{p-1}T_k\right) \to c_k \mu_{\gamma,c} \triangleq c_k M_J(-1) \tag{79}$$



(a)



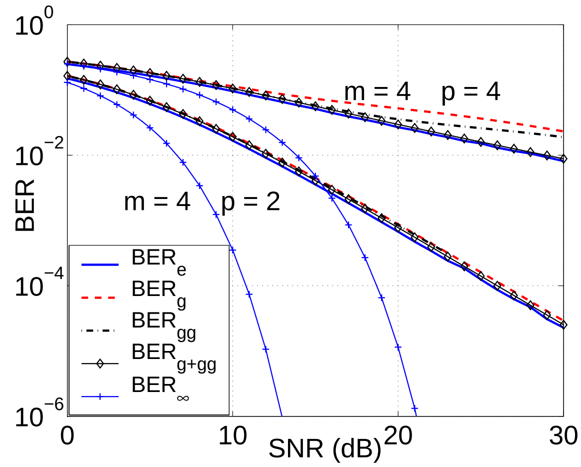
(b)

Fig. 8. BER curves for uncorrelated channels. $\gamma = \frac{1}{2}$ and $\gamma = 1$ are used for computing BER_∞ in both (a) and (b). BER_e stands for the exact BER. BER_g and BER_{gg} are computed by assuming SINR_k is a Gamma and a generalized Gamma, respectively. BER_{g+gg} uses the exact Gamma distribution for S_k and approximates T_k by a generalized Gamma. BER_∞ is the asymptotic BER.

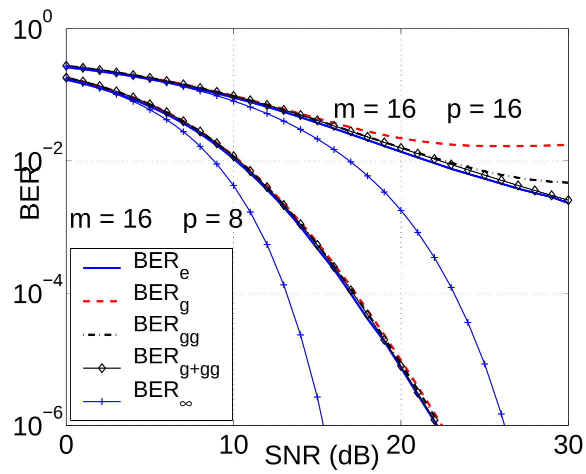
Proof:

$$
\begin{aligned}
&\mathrm{E}\left(\frac{p}{p-1}T_k\right)\\
=&\mathrm{E}\left(\frac{1}{p-1}\mathrm{E}\left(pT_k|\mathbf{H}_{(-k)}\right)\right)\\
=&\mathrm{E}\left(\frac{1}{p-1}\frac{p}{m}\sum_{i=1}^{p-1}\mathrm{E}\left(\|t_{k,i}\|^2\right)\lambda_i\right)\\
=&\Sigma_k\frac{p}{m}\mathrm{E}\left(\frac{\mathrm{tr}\left(\mathbf{\Lambda}\right)}{p-1}\right)\\
\to&\Sigma_k\frac{p}{m}\mathrm{E}\left(M_J(-1)\right)=c_kM_J(-1),
\end{aligned}
$$

by the bounded convergence theorem [26] (Section 1.3b),



(a)



(b)

Fig. 9. BER curves for equicorrelation $\rho = 0.5$. Note that the BER_∞s are different because we did not simulate the true limit of the mean.

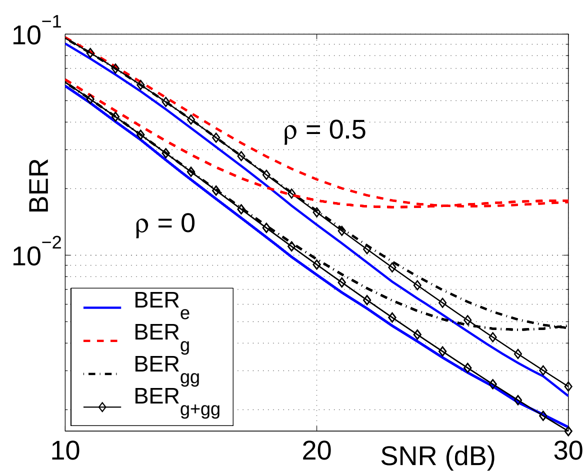


Fig. 10. Compare the uncorrelated and correlated BER curves. $m = 16$, $p = 16$.

because $\frac{\text{tr}(\mathbf{\Lambda})}{p-1} \leq 1$.

■

Restate *lemma* 4:

$$\text{Var}\left(\frac{p}{\sqrt{p-1}}T_k\right) \to c_k^2\sigma_{\gamma,c}^2 \triangleq c_k^2 M_J'(-1). \tag{80}$$

Proof:

$$\begin{aligned}
&\text{Var}\left(\frac{p}{\sqrt{p-1}}T_k\right)\\
=&\frac{p^2}{p-1}\text{Var}\left(\frac{1}{m}\sum_{i=1}^{p-1}\|t_{k,i}\|^2\lambda_i\right)\\
=&\frac{p^2}{p-1}\text{E}\left(\text{Var}\left(\frac{1}{m}\sum_{i=1}^{p-1}\|t_{k,i}\|^2\lambda_i\middle|\mathbf{H}_{(-k)}\right)\right)\\
&+\frac{p^2}{p-1}\text{Var}\left(\text{E}\left(\frac{1}{m}\sum_{i=1}^{p-1}\|t_{k,i}\|^2\lambda_i\middle|\mathbf{H}_{(-k)}\right)\right)\\
=&\frac{p^2}{m^2}\Sigma_k^2\text{E}\left(\frac{\text{tr}\left(\mathbf{\Lambda}^2\right)}{p-1}\right)+\frac{p^2}{m^2}\Sigma_k^2(p-1)\text{Var}\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}\right)\\
\to& c_k^2 M_J'(-1),
\end{aligned}$$

because $\text{E}\left(\frac{\text{tr}(\mathbf{\Lambda}^2)}{p-1}\right) \to M_J'(-1)$ by the bounded convergence theorem, and $(p-1)\text{Var}\left(\frac{\text{tr}(\mathbf{\Lambda})}{p-1}\right) \to 0$, which can be proved using the results from the concentration of spectral measures for random matrices [32]. The result we need is Corollary 1.8b in [32], which can be stated as follows,

$$P\left(\left|\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}-\text{E}\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}\right)\right|>\epsilon\right)\leq 2e^{-\delta(p-1)^2\epsilon^2}, \tag{81}$$

for any $\epsilon > 0$. δ depends on the spectral radius of $\mathbf{R}_{(-k,-k)}$, the logarithmic Sobolev Inequality constant, and the Lipschitz constant of $g(x) = f(x^2) = \frac{1}{1+x^2}$. We can easily check that the function $f(x) = \frac{1}{1+x}$ is convex Lipschitz and $g(x)$ has a finite Lipschitz norm. Therefore,

$$\begin{aligned}
&\text{Var}\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}\right)\\
=&\int_0^\infty 2xP\left(\left|\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}-\text{E}\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}\right)\right|>x\right)dx.\\
\leq&4\int_0^\infty xe^{-\delta(p-1)^2x^2}dx=\frac{2}{\delta(p-1)^2},
\end{aligned} \tag{82}$$

which implies $(p-1)\text{Var}\left(\frac{\text{tr}(\mathbf{\Lambda})}{p-1}\right) \to 0$. This completes the proof.

■

Restate *lemma* 5:

$$\begin{aligned}
\frac{1}{p-1}\text{Sk}(pT_k) = \frac{1}{p-1}\text{E}\left((pT_k-\text{E}(pT_k))^3\right)\\
\to 2c_k^3\eta_{\gamma,c}\triangleq c_k^3M_J''(-1)
\end{aligned} \tag{83}$$

Proof:

$$\begin{aligned}
&\frac{1}{p-1}\text{E}\left((pT_k-\text{E}(pT_k))^3\right)\\
=&\frac{1}{p-1}\text{E}\left(\text{E}\left(\left(pT_k-\text{E}\left(pT_k\middle|\,\mathbf{H}_{(-k)}\right)\right)^3\middle|\,\mathbf{H}_{(-k)}\right)\right)\\
&+\frac{1}{p-1}\text{E}\left(\left(\text{E}\left(pT_k\middle|\,\mathbf{H}_{(-k)}\right)-\text{E}(pT_k)\right)^3\right)\\
&+\frac{3}{p-1}\text{Cov}\left(\text{E}\left(pT_k|\mathbf{H}_{(-k)}\right),\text{Var}\left(pT_k|\mathbf{H}_{(-k)}\right)\right).
\end{aligned} \tag{84}$$

We can show that the first term in the right-hand side of (84) converges to $c_k^3M_J''(-1)$ as follows:

$$\begin{aligned}
&\frac{1}{p-1}\text{E}\left(\text{E}\left(\left(pT_k-\text{E}\left(pT_k\middle|\,\mathbf{H}_{(-k)}\right)\right)^3\middle|\,\mathbf{H}_{(-k)}\right)\right)\\
=&\frac{p^3}{m^3}\frac{1}{p-1}\text{E}\left(\text{E}\left(\left(\sum_{i=1}^{p-1}\left(\|t_{k,i}\|^2-\Sigma_k\right)\lambda_i\right)^3\middle|\,\mathbf{H}_{(-k)}\right)\right)\\
=&\frac{p^3}{m^3}\frac{1}{p-1}\text{E}\left(\sum_{i=1}^{p-1}\text{E}\left(\|t_{k,i}\|^2-\Sigma_k\right)^3\lambda_i^3\right)\\
=&\frac{p^3}{m^3}\frac{1}{p-1}\text{E}\left(\sum_{i=1}^{p-1}2\Sigma_k^3\lambda_i^3\right)\\
=&2\frac{p^3}{m^3}\Sigma_k^3\text{E}\left(\frac{\text{tr}\left(\mathbf{\Lambda}^3\right)}{p-1}\right)\\
\to&2\frac{p^3}{m^3}\Sigma_k^3\frac{1}{2}\text{E}\left(M_J''(-1)\right)=c_k^3M_J''(-1).
\end{aligned} \tag{85}$$

We can show the last two terms in the right-hand side of (84) tend to 0.

Expand

$$\begin{aligned}
&\frac{1}{p-1}\text{E}\left(\left(\text{E}\left(pT_k\middle|\,\mathbf{H}_{(-k)}\right)-\text{E}(pT_k)\right)^3\right)\\
=&\frac{p^3}{m^3}\Sigma_k^3(p-1)^2\text{E}\left(\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}-\frac{\text{E}\left(\text{tr}\left(\mathbf{\Lambda}\right)\right)}{p-1}\right)^3\right).
\end{aligned}$$

Apply the concentration theorem one more time,

$$\begin{aligned}
&\text{E}\left(\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}-\frac{\text{E}\left(\text{tr}\left(\mathbf{\Lambda}\right)\right)}{p-1}\right)^3\right)\\
\leq&\int_0^\infty 3x^2P\left(\left|\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}-\text{E}\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}\right)\right|>x\right)dx\\
\leq&3\int_0^\infty x^2 2e^{-\delta(p-1)^2x^2}dx=\frac{3}{2}\frac{1}{(p-1)^3}\sqrt{\frac{\pi}{\delta^3}},
\end{aligned} \tag{86}$$

which implies $\frac{1}{p-1}\text{E}\left(\left(\text{E}\left(pT_k\middle|\,\mathbf{H}_{(-k)}\right)-\text{E}(pT_k)\right)^3\right) \to 0$.

The last term in (84) also tends to zero, again using the concentration theorem and the following inequality,

$$\begin{aligned}
&\frac{3}{p-1}\text{Cov}\left(\text{E}\left(pT_k|\mathbf{H}_{(-k)}\right),\text{Var}\left(pT_k|\mathbf{H}_{(-k)}\right)\right)\\
=&3\frac{p^3}{m^3}\Sigma_k^3(p-1)\text{Cov}\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1},\frac{\text{tr}\left(\mathbf{\Lambda}^2\right)}{p-1}\right)\\
\leq&3\frac{p^3}{m^3}(p-1)\Sigma_k^3\sqrt{\text{Var}\left(\frac{\text{tr}\left(\mathbf{\Lambda}\right)}{p-1}\right)}\sqrt{\text{Var}\left(\frac{\text{tr}\left(\mathbf{\Lambda}^2\right)}{p-1}\right)}.
\end{aligned} \tag{87}$$

We have shown $\mathrm{Var}\left(\frac{\mathrm{tr}(\mathbf{\Lambda})}{p-1}\right) \le \frac{2}{\delta(p-1)^2}$. Similar arguments will show that $\mathrm{Var}\left(\frac{\mathrm{tr}(\mathbf{\Lambda^2})}{p-1}\right) \le \frac{2}{\delta'(p-1)^2}$, for a different constant δ'. Therefore, the last term in (84) also tends to zero.

Combining the results for the three terms of (84) together completes the proof. ■

Appendix II
Proof of Lemma 6

Restate *Lemma* 6:

$$\tilde{\mu}_p \ge \tilde{\mu}_{p-1} = \tilde{\mu}_{R,p-1} \ge \tilde{\mu}_{R,p}. \tag{88}$$

Proof: Expand $\tilde{\mu}_p$,

$$\tilde{\mu}_p = \frac{1}{2}c\frac{m-p+1}{p} + \frac{1}{2}\frac{cm}{p^2} - \frac{1}{2}\frac{p-1}{p}\left(1-\kappa_p\right) \tag{89}$$

where

$$\kappa_p = \sqrt{c^2\frac{m^2}{p^2}\left(1-\frac{p}{m}\right)^2 + 2c\frac{m}{p}\left(1+\frac{p}{m}\right)+1},$$

$$\begin{aligned}(p\kappa_p)^2 &= (cm+p-cp)^2 + 4cp^2 \\ &= (cm+p)^2 + c^2p^2 + 2cp^2 - 2c^2mp.\end{aligned}$$

Expand $\tilde{\mu}_{p-1}$,

$$\tilde{\mu}_{p-1} = \frac{1}{2}c\frac{m-p+1}{p} - \frac{1}{2}\left(1-\kappa_{p-1}\right) \tag{90}$$

where

$$\kappa_{p-1} = \sqrt{\kappa_p^2 + \frac{c^2}{p^2} + 2c^2\frac{m}{p^2} - 2\frac{c^2}{p} - \frac{2c}{p}},$$

$$(p\kappa_{p-1})^2 = (p\kappa_p)^2 + c^2 + 2c^2m - 2c^2p - 2cp.$$

Therefore,

$$\tilde{\mu}_p - \tilde{\mu}_{p-1} = \frac{1}{2}\left(\frac{cm}{p^2} + \frac{1}{p} + \frac{p-1}{p}\kappa_p - \kappa_{p-1}\right).$$

Suffices to show

$$\begin{aligned}& cm + p + p(p-1)\kappa_p - p^2\kappa_{p-1} \ge 0 \\ \Longleftrightarrow & (cm+p)^2 + (p-1)p^2\kappa_p^2 + 2(cm+p)(p-1)p\kappa_p \\ & \ge p^4\kappa_{p-1}^2 \\ \Longleftrightarrow & (cm+p)(p-1)p\kappa_p \\ & \ge (p-1)(cm+p)^2 - cp(p-1)(cm-p) \\ \Longleftrightarrow & (cm+p)p\kappa_p \ge (cm+p)^2 - cp(cm-p) \\ \Longleftrightarrow & (cm+p)^2 \ge (cm-p)^2,\end{aligned}$$

which is true. Therefore, we prove $\tilde{\mu}_p \ge \tilde{\mu}_{p-1}$. Here we use"$\Longleftrightarrow$" for "is equivalent to".

To show $\tilde{\mu}_{p-1} \ge \tilde{\mu}_{R,p}$, note that

$$\tilde{\mu}_{p-1} - \tilde{\mu}_{R,p} = \frac{1}{2}\left(\frac{c}{p} + \kappa_{p-1} - \kappa_p\right).$$

Suffices to show

$$\begin{aligned}& p\kappa_{p-1} \ge p\kappa_p - c \\ \Longleftrightarrow & p^2\kappa_{p-1}^2 \ge p^2\kappa_p^2 + c^2 - 2cp\kappa_p \\ \Longleftrightarrow & c^2 + 2c^2m - 2c^2p - 2cp \ge c^2 - 2cp\kappa_p \\ \Longleftrightarrow & p\kappa_p \ge -cm + cp + p,\end{aligned}$$

which is true. We complete the proof. ■

Appendix III
Proof of Lemma 7

We will frequently use the following three inequalities:

$$\begin{aligned}Z_k^{(1)} &\triangleq \sum_{i=1}^{p-1}\left(\frac{1}{m}\|z_i\|^2\lambda_i\right) = \sum_{i=1}^{p-1}\frac{\frac{1}{m}\|z_i\|^2}{1+\frac{1}{m}d_i^2} \\ &\le \|\mathbf{R}_{(-k,-k)})^{-1}r_{k(-k)}\|^2 \triangleq e_k.\end{aligned} \tag{91}$$

$$Z_k^{(2)} \triangleq \sum_{i=1}^{p-1}\frac{\frac{1}{m}2\|z_i\|^2}{\left(1+\frac{1}{m}d_i^2\right)^2} \le \frac{1}{2}e_k. \tag{92}$$

$$Z_k^{(3)} \triangleq \sum_{i=1}^{p-1}\frac{\frac{1}{m}6\|z_i\|^2}{\left(1+\frac{1}{m}d_i^2\right)^3} \le \frac{8}{9}e_k. \tag{93}$$

(Note that $\frac{x}{1+x} \le 1$, $\frac{2x}{(1+x)^2} \le \frac{1}{2}$, $\frac{6x}{(1+x)^3} \le \frac{8}{9}$, for any $x \ge 0$.)

We begin with the first moment. Conditional on $\mathbf{H}_{(-k)}$,

$$\begin{aligned}\mathrm{E}(T_k|\,\mathbf{H}_{(-k)}) =& \frac{1}{m}\sum_{i=1}^{p-1}\frac{\Sigma_k + \|z_i\|^2}{1+\frac{1}{m}d_i^2} \\ =& \frac{1}{m}\Sigma_k\mathrm{tr}(\mathbf{\Lambda}) + Z_k^{(1)} \\ \le& \frac{1}{m}\Sigma_k\mathrm{tr}(\mathbf{\Lambda}) + e_k.\end{aligned} \tag{94}$$

Recall $\frac{\mathrm{tr}(\mathbf{\Lambda})}{p-1} \xrightarrow{P} M_J(-1)$, the limiting upper bound of $\mathrm{E}(T_k)$ is then given by

$$\mathrm{E}(T_k)^U = \frac{p-1}{m}\Sigma_k M_J(-1) + e_k. \tag{95}$$

We can derive the limiting upper bound for the variance of T_k by following the proof of Lemma 4.

We know

$$\begin{aligned}\mathrm{Var}\left(\frac{p}{\sqrt{p-1}}T_k\right) =& \frac{p^2}{p-1}\mathrm{E}\left(\frac{1}{m^2}\sum_{i=1}^{p-1}\frac{\Sigma_k^2 + 2\|z_i\|^2\Sigma_k}{\left(1+\frac{1}{m}d_i^2\right)^2}\right) \\ &+ \frac{p^2}{p-1}\mathrm{Var}\left(\frac{1}{m}\sum_{i=1}^{p-1}\frac{\Sigma_k + \|z_i\|^2}{1+\frac{1}{m}d_i^2}\right).\end{aligned} \tag{96}$$

The first term in the right-hand side of (96) can be written as

$$\begin{aligned}&\frac{p^2}{m^2}\Sigma_k^2\mathrm{E}\left(\frac{\mathrm{tr}(\mathbf{\Lambda}^2)}{p-1}\right) + \frac{p^2}{m}\frac{1}{p-1}\Sigma_k\mathrm{E}(Z_k^{(2)}) \\ \le&\frac{p^2}{m^2}\Sigma_k^2 M_J''(-1) + \frac{p^2}{m}\frac{1}{p-1}\Sigma_k\frac{1}{2}e_k,\end{aligned} \tag{97}$$

in the limiting sense.

Using the fact that $\mathrm{Var}(x+y) \le \mathrm{Var}(x) + \mathrm{E}(y^2) + 2\sqrt{\mathrm{Var}(x)}\sqrt{\mathrm{E}(y^2)}$, we can bound the second term in (96) by

$$\begin{aligned}&\frac{p^2}{m^2}(p-1)\Sigma_k^2\mathrm{Var}\left(\frac{\mathrm{tr}(\mathbf{\Lambda})}{p-1}\right) + \frac{p^2}{p-1}\mathrm{E}\left(Z_k^{(1)}\right)^2 \\ &+ 2\frac{p^2}{m}\Sigma_k\sqrt{\mathrm{Var}\left(\frac{\mathrm{tr}(\mathbf{\Lambda})}{p-1}\right)}\sqrt{\mathrm{E}\left(Z_k^{(1)}\right)^2} \\ \le&\frac{p^2}{m^2}\frac{2\Sigma_k^2}{\delta(p-1)} + \frac{p^2}{p-1}e_k^2 + 2\frac{p^2}{m(p-1)}\Sigma_k\sqrt{\frac{2}{\delta}}e_k,\end{aligned} \tag{98}$$

where δ is the same constant in Lemma 4.

Combining (97), (98), and ignoring the higher order term in (98), we can get a comprehensive upper bound

$$\mathrm{Var}\left(\frac{p}{\sqrt{p-1}}T_k\right)^U = \frac{p^2}{m^2}\Sigma_k^2 M_J'(-1) + \frac{p^2}{m}\Sigma_k\frac{1}{p-1}\frac{1}{2}e_k + \frac{p^2}{p-1}e_k^2 + 2\frac{p^2}{m(p-1)}\Sigma_k\sqrt{\frac{2}{\delta}}e_k, \tag{99}$$

which, for convenience, is written as

$$\mathrm{Var}\left(T_k\right)^U = \frac{p-1}{m^2}\Sigma_k^2 M_J'(-1) + \frac{1}{m}\Sigma_k\left(\frac{1}{2}+2\sqrt{\frac{2}{\delta}}\right)e_k + e_k^2. \tag{100}$$

We can get the limiting upper bound for the third central moment by following the proof of Lemma 5.

Recall

$$\begin{aligned}&\frac{p^3}{p-1}\mathrm{Sk}\left(T_k\right)\\ =&\frac{1}{p-1}\mathrm{E}\left(\mathrm{E}\left(\left(pT_k - \mathrm{E}\left(pT_k|\,\mathbf{H}_{(-k)}\right)\right)^3\Big|\,\mathbf{H}_{(-k)}\right)\right)\\ &+\frac{1}{p-1}\mathrm{E}\left(\left(\mathrm{E}\left(pT_k|\,\mathbf{H}_{(-k)}\right)-\mathrm{E}(pT_k)\right)^3\right)\\ &+\frac{3}{p-1}\mathrm{Cov}\left(\mathrm{E}\left(pT_k|\mathbf{H}_{(-k)}\right),\mathrm{Var}\left(pT_k|\mathbf{H}_{(-k)}\right)\right).\end{aligned} \tag{101}$$

Expand the first term of the right-hand side of (101) to get

$$\begin{aligned}&\frac{p^3}{m^3}\Sigma_k^3\mathrm{E}\left(\frac{\mathrm{tr}(\mathbf{\Lambda}^3)}{p-1}\right)+\frac{p^3}{m^2}\frac{1}{p-1}\Sigma_k^2 Z_k^{(3)}\\ \le&\frac{p^3}{m^3}\Sigma_k^3 M_J''(-1)+\frac{p^3}{m^2}\frac{1}{p-1}\Sigma_k^2\frac{8}{9}e_k.\end{aligned} \tag{102}$$

Expanding the second term of (101) and selectively ignoring some negative terms, we can get a bound

$$\begin{aligned}&\frac{1}{p-1}\mathrm{E}\left(\left(\mathrm{E}\left(pT_k|\,\mathbf{H}_{(-k)}\right)-\mathrm{E}(pT_k)\right)^3\right)\\ \le&\frac{p^3}{m^3}\frac{1}{p-1}\mathrm{E}\left(\left(\mathrm{tr}\left(\mathbf{\Lambda}\right)-\mathrm{E}\left(\mathrm{tr}\left(\mathbf{\Lambda}\right)\right)\right)\Sigma_k + me_k\right)^3\\ \le&\frac{p^3}{m^3}\Sigma_k^3\left(\frac{3}{2}\frac{1}{p-1}\sqrt{\frac{\pi}{\delta^3}}\right)+\frac{p^3}{p-1}e_k^3+3\frac{p^3}{m^2}\Sigma_k^2\left(\frac{2}{\delta(p-1)}\right)e_k.\end{aligned}$$

We can also bound the last term of (101) by

$$\begin{aligned}&\frac{3}{p-1}\mathrm{Cov}\left(\mathrm{E}\left(pT_k|\mathbf{H}_{(-k)}\right),\mathrm{Var}\left(pT_k|\mathbf{H}_{(-k)}\right)\right)\\ \le&\frac{3}{p-1}\sqrt{\mathrm{Var}\left(\mathrm{E}\left(pT_k|\mathbf{H}_{(-k)}\right)\right)}\sqrt{\mathrm{Var}\left(\mathrm{Var}\left(pT_k|\mathbf{H}_{(-k)}\right)\right)}.\end{aligned}$$

We know

$$\mathrm{Var}\left(\mathrm{E}\left(pT_k|\mathbf{H}_{(-k)}\right)\right)\le p^2e_k^2+2\frac{p^2}{m}\Sigma_k\sqrt{\frac{2}{\delta}}e_k, \tag{103}$$

and

$$\begin{aligned}&\mathrm{Var}\left(\mathrm{Var}\left(pT_k|\mathbf{H}_{(-k)}\right)\right)\\ =&\frac{p^4}{m^4}\Sigma_k^2\mathrm{Var}\left(\Sigma_k\mathrm{tr}\left(\mathbf{\Lambda}^2\right)+mZ_k^{(2)}\right)\\ \le&\frac{p^4}{m^4}\Sigma_k^2\left(\Sigma_k^2\frac{2}{\delta'}+m^2\left(\frac{1}{2}e_k\right)^2+\Sigma_k me_k\sqrt{\frac{2}{\delta'}}\right)\end{aligned}$$

Combing all the bounds and limits, we can obtain a limiting upper bound for the third moment

$$\begin{aligned}\mathrm{Sk}(T_k)^U =& \frac{p-1}{m^3}\Sigma_k^3 M_J''(-1)+\frac{8/9}{m^2}\Sigma_k^2 e_k+\frac{6}{m^2}\Sigma_k^2\frac{1}{\delta}e_k+e_k^3\\ &+\frac{3}{2}e_k\Sigma_k\frac{1}{m}\sqrt{e_k+2\frac{1}{m}\Sigma_k\sqrt{\frac{2}{\delta}}}\sqrt{e_k+2\frac{1}{m}\Sigma_k\sqrt{\frac{2}{\delta'}}}.\end{aligned} \tag{104}$$

APPENDIX IV
PROOF OF LEMMA 9

Restate *Lemma* 9

$$W=\frac{mT_k-\Sigma_k\mathrm{tr}(\mathbf{\Lambda})}{\sqrt{p-1}}\overset{D}{\Longrightarrow}\mathrm{N}\left(0,\Sigma_k^2\sigma_{\gamma,c}^2\right) \tag{105}$$

Proof: It suffices to show that the characteristic function of W, $\mathrm{E}\left(e^{jyW}\right)\to\exp\left(-\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right)$, where

$$\mathrm{E}\left(e^{jyW}\right)=\mathrm{E}\left(\mathrm{E}\left(e^{jyW}|\mathbf{H}_{(-k)}\right)\right),\ j=\sqrt{-1}. \tag{106}$$

Conditional on $\mathbf{H}_{(-k)}$,

$$\begin{aligned}&\mathrm{E}\left(e^{jyW}|\mathbf{H}_{(-k)}\right)\\ =&\mathrm{E}\left(\exp\left(jy\frac{mT_k-\Sigma_k\mathrm{tr}(\mathbf{\Lambda})}{\sqrt{p-1}}\right)\Big|\,\mathbf{H}_{(-k)}\right)\\ =&\exp\left(\frac{-jy\Sigma_k\mathrm{tr}(\mathbf{\Lambda})}{\sqrt{p-1}}\right)\prod_{i=1}^{p-1}\mathrm{E}\left(\exp\left(jy\|t_{k,i}\|^2\lambda_i\right)\right)\\ =&\exp\left(\frac{-jy\Sigma_k\mathrm{tr}(\mathbf{\Lambda})}{\sqrt{p-1}}\right)\\ &\prod_{i=1}^{p-1}\left(1-\frac{1}{\sqrt{p-1}}jy\Sigma_k\lambda_i\right)^{-1}\exp\left(\|z_i\|^2\frac{\frac{jy\lambda_i}{\sqrt{p-1}}}{1-\frac{jy\Sigma_k\lambda_i}{\sqrt{p-1}}}\right)\\ =&\exp\left(\frac{-jy\Sigma_k\mathrm{tr}(\mathbf{\Lambda})}{\sqrt{p-1}}\right)L_1(p)L_2(p).\end{aligned} \tag{107}$$

Note that $\lambda_i=\frac{1}{1+\frac{1}{m}d_i^2}\le 1$. For any fixed $y\in\mathbb{R}$, since $p\to\infty$, we can assume $\frac{|y|}{\sqrt{p-1}}<1$, $\frac{|y|\Sigma_k\lambda_i}{\sqrt{p-1}}<1$, which allows us to conduct complex series expansions.

We can write

$$\begin{aligned}L_2(p)=&\prod_{i=1}^{p-1}\exp\left(\|z_i\|^2\frac{\frac{jy\lambda_i}{\sqrt{p-1}}}{1-\frac{jy\Sigma_k\lambda_i}{\sqrt{p-1}}}\right)\\ =&\exp\left(\sum_{i=1}^{p-1}\|z_i\|^2\frac{\frac{jy\lambda_i}{\sqrt{p-1}}}{1-\frac{jy\Sigma_k\lambda_i}{\sqrt{p-1}}}\right).\end{aligned} \tag{108}$$

Then

$$\begin{aligned}\|\log(L_2(p))\|\le&\sum_{i=1}^{p-1}\left\|\|z_i\|^2\frac{\frac{jy\lambda_i}{\sqrt{p-1}}}{1-\frac{jy\Sigma_k\lambda_i}{\sqrt{p-1}}}\right\|\\ \le&\sum_{i=1}^{p-1}\|z_i\|^2\frac{|y|\lambda_i}{\sqrt{p-1}}2\\ \le&2|y|\frac{m}{\sqrt{p-1}}Z_k^{(1)}\\ \le&2|y|\frac{m}{\sqrt{p-1}}e_k\to 0,\end{aligned} \tag{109}$$

because of the assumption that $e_k \to 0$ faster than $O(p^{-\frac{1}{2}})$. Therefore $L_2(p) \xrightarrow{P} 1$, hence can be ignored.

$$\begin{aligned}
L_1(p) =& \prod_{i=1}^{p-1}\left(1-\frac{1}{\sqrt{p-1}}jy\Sigma_k\lambda_i\right)^{-1}\\
=&\exp\left(-\sum_{i=1}^{p-1}\log\left(1-\frac{jy\Sigma_k\lambda_i}{\sqrt{p-1}}\right)\right)\\
=&\exp\left(\sum_{i=1}^{p-1}\sum_{n=1}^{\infty}\frac{(jy\Sigma_k\lambda_i)^n}{n(p-1)^{n/2}}\right)\\
=&\exp\left(\sum_{n=1}^{\infty}\sum_{i=1}^{p-1}\frac{(jy\Sigma_k\lambda_i)^n}{n(p-1)^{n/2}}\right)\\
=&\exp\left(\sum_{i=1}^{p-1}\frac{jy\Sigma_k\lambda_i}{(p-1)^{1/2}}\right)\exp\left(\sum_{i=1}^{p-1}\frac{-y^2\Sigma_k^2\lambda_i^2}{2(p-1)}\right)\times\\
&\exp\left(\sum_{n=3}^{\infty}\sum_{i=1}^{p-1}\frac{(jy\Sigma_k\lambda_i)^n}{n(p-1)^{n/2}}\right)\\
=&\exp\left(\frac{jy\Sigma_k\mathrm{tr}(\mathbf{\Lambda})}{\sqrt{p-1}}\right)\exp\left(\frac{-y^2\Sigma_k^2\mathrm{tr}(\mathbf{\Lambda}^2)}{2(p-1)}\right)L_3(p). \quad (110)
\end{aligned}$$

We will show that $L_3(p)$ converges to 1 in probability. Note that in the above derivations, we can expand the complex logarithm and switch the order of summations because of the condition $\frac{|y|}{\sqrt{p-1}} < 1$.

$$\begin{aligned}
\|\log(L_3(p))\| =& \left\|\sum_{n=3}^{\infty}\sum_{i=1}^{p-1}\frac{(jy\Sigma_k\lambda_i)^n}{n(p-1)^{n/2}}\right\|\\
\leq& \sum_{n=3}^{\infty}\sum_{i=1}^{p-1}\frac{(|y|\Sigma_k)^n}{n(p-1)^{n/2}}\\
\leq& \sum_{n=3}^{\infty}\frac{p-1}{3}\frac{(|y|\Sigma_k)^n}{(p-1)^{n/2}}\\
=& \frac{p-1}{3}\frac{\frac{|y|^3\Sigma_k^3}{(p-1)^{3/2}}}{1-\frac{|y|\Sigma_k}{(p-1)^{1/2}}}\xrightarrow{P}0, \quad \text{i.e., } L(p,c)\xrightarrow{P}1.
\end{aligned}$$

After cancellations,

$$\begin{aligned}
\mathrm{E}\left(e^{jyW}|\mathbf{H}_{(-k)}\right) \xrightarrow{P}& \lim_{p\to\infty}\exp\left(\frac{-y^2\Sigma_k^2\mathrm{tr}(\mathbf{\Lambda}^2)}{2(p-1)}\right)\\
=&\exp\left(-\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right), \quad (111)
\end{aligned}$$

which is the characteristic function of a Normal random variable with mean 0 and variance $\Sigma_k^2\sigma_{\gamma,c}^2$.

To show $\mathrm{E}\left(e^{jyW}\right) \to \exp\left(-\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right)$, it suffices to show $\left|\mathrm{E}\left(\exp\left(jyW+\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right)\right)-1\right| \to 0$.

$$\begin{aligned}
&\left|\mathrm{E}\left(\exp\left(jyW+\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right)\right)-1\right|\\
=&\mathrm{E}\left|\mathrm{E}\left(\exp\left(jyW+\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right)\middle|\mathbf{H}_{(-k)}\right)-1\right|\\
=&\mathrm{E}\left|\exp\left(\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right)\mathrm{E}\left(\exp(jyW)|\,\mathbf{H}_{(-k)}\right)-1\right|\\
\to&\mathrm{E}\left|\exp\left(\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right)\exp\left(-\frac{y^2}{2}\Sigma_k^2\sigma_{\gamma,c}^2\right)-1\right| = 0.
\end{aligned}$$

by the bounded convergence theorem. This completes the proof. ■

Appendix V
Proof of Lemma 10

First we can show that $RS_{T_k} = \frac{\sigma_{\gamma,c}^4}{\mu_{\gamma,c}\eta_{\gamma,c}} \leq 1$, i.e., $\sigma_{\gamma,c}^4 \leq \mu_{\gamma,c}\eta_{\gamma,c}$. To simplify the expressions, let $\sigma_{\gamma,c}^2 = \mu_{\gamma,c} - O$, and $\eta_{\gamma,c} = \sigma_{\gamma,c}^2 - Q = \mu_{\gamma,c} - O - Q$, where $O = \frac{\tilde{c}(1+\gamma)+1-\kappa}{2\tilde{c}\gamma\kappa}$, $Q = \frac{\tilde{c}}{\kappa^3}$. Then

$$\begin{aligned}
\mu_{\gamma,c}\eta_{\gamma,c} - \sigma_{\gamma,c}^4 =& \mu_{\gamma,c}\left(\sigma_{\gamma,c}^2 - Q\right) - \sigma_{\gamma,c}^4\\
=&\sigma_{\gamma,c}^2 O - \mu_{\gamma,c}Q, \quad (112)
\end{aligned}$$

$$\begin{aligned}
\sigma_{\gamma,c}^2 =& \mu_{\gamma,c} - \frac{\tilde{c}(1+\gamma)+1-\kappa}{2\tilde{c}\gamma\kappa}\\
=&\frac{\left(\tilde{c}(1-\gamma)^2+(1+\gamma)-(1-\gamma)\kappa\right)}{2\gamma\kappa}, \quad (113)
\end{aligned}$$

$$\begin{aligned}
\sigma_{\gamma,c}^2 O =& \frac{\left(\tilde{c}(1-\gamma)^2+(1+\gamma)-(1-\gamma)\kappa\right)}{2\gamma\kappa}\frac{\left(\tilde{c}(1+\gamma)+1\right)-\kappa}{2\tilde{c}\gamma\kappa}\\
=& \frac{1}{2\tilde{c}\gamma^2\kappa^2}\left(\kappa^2-\kappa\left(\tilde{c}(1-\gamma)+1\right)-2\tilde{c}\gamma\right), \quad (114)
\end{aligned}$$

$$\mu_{\gamma,c}Q = \frac{\kappa-\tilde{c}(1-\gamma)-1}{2\tilde{c}\gamma}\frac{\tilde{c}}{\kappa^3} = \frac{\kappa-\tilde{c}(1-\gamma)-1}{2\gamma\kappa^3}. \quad (115)$$

Plug (114) and (115) into (112),

$$\begin{aligned}
&\mu_{\gamma,c}\eta_{\gamma,c} - \sigma_{\gamma,c}^4\\
=&\frac{1}{2\tilde{c}\gamma^2\kappa^3}\left(\kappa^3-\kappa^2(\tilde{c}(1-\gamma)+1)-3\tilde{c}\gamma\kappa+\tilde{c}^2\gamma(1-\gamma)+\tilde{c}\gamma\right)
\end{aligned} \quad (116)$$

Therefore,

$$\begin{aligned}
&\mu_{\gamma,c}\eta_{\gamma,c} - \sigma_{\gamma,c}^4 \geq 0\\
\Longleftrightarrow&\kappa^3-\kappa^2(\tilde{c}(1-\gamma)+1)-3\tilde{c}\gamma\kappa+\tilde{c}^2\gamma(1-\gamma)+\tilde{c}\gamma \geq 0\\
\Longleftrightarrow&\kappa(\kappa^2-3\tilde{c}\gamma) \geq (\tilde{c}(1-\gamma)+1)\left(\kappa^2-\tilde{c}\gamma\right)\\
\Longleftrightarrow&\kappa^2\left(\kappa^2-3\tilde{c}\gamma\right)^2 \geq \left(\kappa^2-4\tilde{c}\gamma^2\right)\left(\kappa^2-\tilde{c}\gamma\right)^2\\
\Longleftrightarrow&\kappa^2\left(2\tilde{c}\gamma\right)\left(2\kappa^2-4\tilde{c}\gamma\right) \leq 4\tilde{c}\gamma\left(\kappa^2-\tilde{c}\gamma\right)^2\\
\Longleftrightarrow&2\tilde{c}^2\gamma^2 \geq 0,
\end{aligned}$$

which is true. It is easy to check to make sure $\kappa^2 - 3\tilde{c}\gamma \geq 0$. This completes the proof for $RS_{T_k} = \frac{\sigma_{\gamma,c}^4}{\mu_{\gamma,c}\eta_{\gamma,c}} \leq 1$, with equality held when $c = 0$ or $\gamma = 0$.

It is easy to show $RS_{\text{SINR}_k} \leq 1$ by noting that $\sigma^2_{\gamma,c} \leq \sqrt{\mu_{\gamma,c}\eta_{\gamma,c}} \leq \frac{\mu_{\gamma,c}+\eta_{\gamma,c}}{2}$, which implies

$$\begin{aligned}
&RS_{\text{SINR}_k}\\
\leq&\frac{\left(1-\frac{m-1}{p}+\frac{p-1}{m}\sqrt{\mu_{\gamma,c}\eta_{\gamma,c}}\right)^2}{\left(1-\frac{p-1}{m}+\frac{p-1}{m}\mu_{\gamma,c}\right)\left(1-\frac{p-1}{m}+\frac{p-1}{m}\eta_{\gamma,c}\right)}\\
=&\frac{(1-\frac{p-1}{m})^2+2\frac{p-1}{m}(1-\frac{p-1}{m})\sqrt{\mu_{\gamma,c}\eta_{\gamma,c}}+\left(\frac{p-1}{m}\right)^2\mu_{\gamma,c}\eta_{\gamma,c}}{(1-\frac{p-1}{m})^2+(1-\frac{p-1}{m})\frac{p-1}{m}\left(\mu_{\gamma,c}+\eta_{\gamma,c}\right)+\left(\frac{p-1}{m}\right)^2\mu_{\gamma,c}\eta_{\gamma,c}}\\
\leq&1,
\end{aligned}$$

with equality held only when $RS_{T_k} = 1$ holds.

The remaining task is to show that $RS_{T_k} \leq RS_{\text{SINR}_k}$ in the limit. In the limit, we can replace $\frac{p-1}{m}$ by γ, thus

$$\begin{aligned}
&RS_{T_k} \leq RS_{\text{SINR}_k}\\
\Longleftrightarrow&\frac{(1-\gamma)^2+2\gamma(1-\gamma)\sigma^2_{\gamma,c}+\gamma^2\sigma^4_{\gamma,c}}{(1-\gamma)^2+\gamma(1-\gamma)\left(\mu_{\gamma,c}+\eta_{\gamma,c}\right)+\gamma^2\left(\mu_{\gamma,c}\eta_{\gamma,c}\right)}\\
&\geq\frac{\sigma^4_{\gamma,c}}{\mu_{\gamma,c}\eta_{\gamma,c}}\\
\Longleftrightarrow&(1-\gamma)\left(\mu_{\gamma,c}\eta_{\gamma,c}-\sigma^4_{\gamma,c}\right)\\
&\geq\gamma\sigma^2_{\gamma,c}\left(\sigma^2_{\gamma,c}\left(\mu_{\gamma,c}+\eta_{\gamma,c}\right)-2\mu_{\gamma,c}\eta_{\gamma,c}\right),
\end{aligned}$$

which is equivalent to $\tilde{c}^2\gamma^2 \geq 0$, after pages of tedious algebra. This completes the proof for $RS_{T_k} \leq RS_{\text{SINR}_k}$, in the limit. See the technical report [23] for details.

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