Instructions: Work ALL problems. This is a closed book/notes test. You may use a blue book or loose note papers. Make sure you enter your name and ID on the cover page of your blue book, (on the top right hand corner of every page if you use loose papers). You are expected to follow rules conforming to UCD regulations. Turn in both your answers and the test copy at the end of the test period. No calculators, computers, cell phones, and other electronic equipments (except watches to tell time) are allowed. Show all details for full/partial credit.

1. Toss a fair coin 4 times. Consider events
   
   \[ A = \text{exactly 2 heads in 4 tosses,} \quad B = \text{"heads" occurs for the second time on the third toss.} \]

   a) Compute \( P(A), P(B) \)
   
   b) Compute \( P(A|B), P(B|A). \)
   
   c) Are \( A \) and \( B \) independent? Provide reason(s).

2. Suppose \( E, F \) and \( G \) are jointly independent events. Show that \( E \) is independent of \( F \cup G \).
   
   a) Let \( X \) be a r.v. with finite variance and let \( a \) be a real number. Show that
   
   \[ E[(X - a)^2] = \text{Var}(X) + [E(X) - a]^2. \]

3. The Starship Enterprise is planning a surprise attack against the Klingons in a neutral quadrant. Possible interference by the Romulans, though, is causing Captain Kirk and Mr. Spock to reassess their strategy. According to Spock's calculations, the probability of the Romulans joining forces with the Klingons is 0.2384. Captain Kirk feels that the probability of the attack's being a success is 0.8 if the Enterprise can catch the Klingons alone but only 0.3 if they have to engage both adversaries. Spock claims that the attack would be a tactical misadventure if its probability of success were not at least 0.7306.
   
   a) Compute \( P(\text{successful attack}) \).
   
   Should Captain Kirk order an attack?

   b) Given that the attack was a failure, what is the conditional probability that the Klingons were alone?
1. \( A = \{ \text{HHTT, HTHT, etc.} \}, \ (4) = 6 \text{ outcomes} \)
\( B = \{ \text{HTHT, HTHH, THHT, THHH} \} \)

(c) \( P(A) = \frac{6}{2^4} = \frac{6}{16}, \ \ P(B) = \frac{4}{16} \)

(b) \( A \cap B = \{ \text{HTHT, THHT} \} \)

\[ P(A|B) = \frac{2/16}{4/16} = \frac{1}{2} \]

\[ P(B|A) = \frac{2/16}{6/16} = \frac{1}{3} \]

(c) Since \( P(A|B) \neq P(A) \), \( A \) and \( B \) are not independent, hence \( A \) and \( B' \) are not independent.

2. (a) \( P(E | F \cup G) = P([E | F] \cup [E | G]) \)

\[ \text{(incl/excl rule)} \Rightarrow P(E | F) + P(E | G) - P(E | F \cap G) \]

\[ \text{(by joint indep.)} \Rightarrow P(E | F) P(F) + P(E | G) P(G) - P(E | F \cap G) P(F \cap G) \]

\[ \text{(by indep.)} \Rightarrow P(E | F) P(F) + P(E | G) P(G) - P(F \cap G) \]

\[ \text{(incl/excl rule)} \Rightarrow P(E | F) P(F) + P(E | G) P(G) - P(F \cap G) \]

(c10) \( \text{LHS} = E[(8 - \mu)^2] = E[8^2 - 2\mu \cdot 8 + \mu^2] \)

\[ = E(8^2) - 2\mu \cdot E(8) + \mu^2 = 8^2 - 2\mu^2 + \mu^2 = 8^2 - 2\mu^2 + \mu^2 = \text{LHS} \]

3. (a) \( P(\text{success or attach}) = P(\text{success} \wedge \text{Klinger alone} + P(\text{success} \wedge \{K + R\}) \)

\[ = P(\text{A} | K) P(K) + P(\text{A} | K + R) P(K + R) \)

\[ = (0.8)(1 - 0.2384) + (0.3)(0.2384) \]

\[ = 0.6098 + 0.0715 \]

\[ = 0.6813 < 0.7306 \text{ : no attach.} \]
\[ 3. \quad \Pr(\text{K alone | failure}) = \frac{\Pr(\text{failure} | K) \Pr(K)}{\Pr(\text{failure})} \]

\[ = \frac{(0.2)(1-0.2384)}{1 - 0.6808} \]

\[ = \frac{0.15232}{0.3192} \]

\[ = 0.4772 \]