Do all problems. Details should be given so that full or part credit can be assigned.

Points

1. A random variable $X$ has density

$$f(x) = \begin{cases} 
  c\sqrt{x} & \text{for } 1 \leq x \leq 4 \\
  0 & \text{otherwise.}
\end{cases}$$

4

(a) Determine the constant $c$.

5

(b) Find the cumulative distribution function (cdf) $F(x)$ of $X$ and graph it.

5

(c) Obtain 10 simulated observations from this model. Denote the 10 random numbers from Uniform(0,1) by $U_1, \ldots, U_{10}$. Indicate step by step your procedure for obtaining the 10 simulated observations from $F$ without actually carrying out the calculations.

2. For each of the following moment generating functions $\phi(t)$ of some r.v. $X$,

(i) $\phi(t) = (3-2e^t)^{-2} \cdot e^{\frac{3}{2}t}, \quad t < \ln(\frac{3}{2})$

(ii) $\phi(t) = \begin{cases} 
  (e^{t/2} - e^{-t/2})/t & \text{if } t \neq 0 \\
  1 & \text{if } t = 0
\end{cases}$

(iii) $\phi(t) = \exp \{t^2\}$

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(a) Identify the model and the parameter(s),

6

(b) Find $EX$;

6

(c) Compute $P(X > 0)$.

(Note: no calculations are necessary for (b) and (c)).
Points

3. Let $X$ be a $N(3,4)$ r.v.

(a) Evaluate $P(1 \leq X \leq 5)$ with the help of the normal table.

(b) Writing $P(1 \leq X \leq 5) = P(|X - 3| \leq 2)$, assess this probability with the help of Chebyshev's inequality. How does this crude assessment compared with the exact answer in (a)?

(c) Find the value $c$ so that $P(X \leq c) = 0.025$.

4. (a) The probability that a high precision resistor manufactured by a certain process outside tolerance limit is 0.01. You have bought 100 of these resistors. What is the probability that 3 or more of these resistors are outside tolerance limits? (Use Poisson approximation.)

4. (b) When a certain light signal pulse falls on a photoelectric detector, the number of electrons emitted and counted is assumed to be a Poisson r.v. with mean $\lambda = 150$. Because of strong background noise, the signal will be declared present only if 140 or more electrons are counted. What is the probability that the signal will be missed? (Use central limit theorem approximation, you may choose not to apply continuity correction.)
5. We want to measure the lengths of 2 rods by using 2 measurements only. 2 methods are proposed as indicated in the following:

**Method 1:** 2 independent measurements X and Y of the 2 rods are made, one one for each rod.

```
  _________________     _________________
  |                |     |                |
  |     measurement X     |     | measurement Y |
  |____________________|     |____________________|

rod 1

rod 2
```

**Method 2:** 2 independent measurements are made according to the configuration below:

```
  _________________     _________________
  |                |     |                |
  |     measurement S (sum)     |     | measurement D (difference) |
  |____________________|     |____________________|

rod 1

rod 2
```

Now assume the measurements are random variables such that the expectation of each gives the true value, and that each measurement has the same variance, i.e., let the true lengths of the 2 rods be $\xi_1$ and $\xi_2$ respectively, and denote the variance by $\sigma^2$, then $EX = \xi_1$, $EY = \xi_2$, $ES = \xi_1 + \xi_2$, $ED = \xi_1 - \xi_2$, $\text{Var}(X) = \text{Var}(Y) = \text{Var}(S) = \text{Var}(D) = \sigma^2$.

4 (a) Provide estimates of $\xi_1$ and $\xi_2$ based on $S$ and $D$. Denote them by $\hat{\xi}_1$ and $\hat{\xi}_2$ respectively.

2 (b) Compute $E\hat{\xi}_1$, $E\hat{\xi}_2$.

8 (c) Compute $\text{Var}(\hat{\xi}_1)$, $\text{Var}(\hat{\xi}_2)$.

2 (d) If we use variance as a measure of performance, which method should we go for?