Midterm 1 Solution

1.

(a)
U and V are independent since $\text{Cov}(U,V) = 0$ and the joint distribution of U and V are multivariate normal. U_1 and V_1 are independent and so are $U_1 - U_3$ and V_2 since U and V are independent.

(b)

$$U - V \sim N_3\left(\mu_U - \mu_V, \Sigma_U + \Sigma_V\right) = N_3\left(\begin{pmatrix}-1\\-2\\0\end{pmatrix}, \begin{pmatrix}7 & 4 & 1\\4 & 8 & 3\\1 & 3 & 6\end{pmatrix}\right)$$

(c)

$$X \sim N_2\left(A\mu_U, A\Sigma_U A^T\right) = N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}8 & 0\\0 & 4\end{pmatrix}\right)$$

(d)
From (c), $\Sigma_X = \begin{pmatrix}8 & 0\\0 & 4\end{pmatrix}$. We can get that the eigenvalues of Σ_X are 8 and 4 by solving $|\Sigma_X - \lambda I| = 0$. The generalized variance of Σ_X is $|\Sigma_X| = 32$.

(e)
$Y = U_1 + U_2 + U_3$ can be written as $Y = c^T U$, where $c^T = (1,1,1)$. So, $\text{Cov}(X,Y) = A\Sigma_U c = \begin{pmatrix}1\\3\end{pmatrix}$

2.

(a)
$H_0 : \mu_X^T = \begin{pmatrix}182 & 150 & 182 & 150\end{pmatrix}$
$H_1 : \mu_X^T \neq \begin{pmatrix}182 & 150 & 182 & 150\end{pmatrix}$

$$\begin{aligned}T^2 =& n(\bar{X} - \mu_X)^T \Sigma_X^{-1} (\bar{X} - \mu_X)\\ =& 25\begin{pmatrix}3.72 & 1.12 & 1.84 & -0.76\end{pmatrix}\Sigma_X^{-1}\begin{pmatrix}3.72\\1.12\\1.84\\-0.76\end{pmatrix}\\ =& 14.0801 > \frac{(n-1)p}{n-p}F_{p,n-p}(0.05) = 12.9829.\end{aligned}$$

Thus, we can reject H_0 at level 0.05.

(b)

$H_0 : \mu_1 = \mu_3$ and $\mu_2 = \mu_4$
H_1 : At least one equality in H_0 does not hold.

$$T^2 = n(c\bar{X})^T(c\Sigma_X c^T)^{-1}(c\bar{X}) = 3.7629 < \frac{(n-1)p}{n-p}F_{p,n-p}(0.05) = 7.1374$$

where $n = 25,\ p = 2,\ c = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$.

$$c\Sigma_X c^T = \begin{pmatrix} 54.5060 & 11.5050 \\ 11.5050 & 28.1060 \end{pmatrix},\ (c\Sigma_X c^T)^{-1} = \begin{pmatrix} 0.0201 & -0.0082 \\ -0.0082 & 0.0389 \end{pmatrix}$$

Thus, we can not reject H_0 at level 0.05.

(c): 95% Bonferroni simultaneous C.I.:

$(\mu_1 - \mu_3)$:

$$(\bar{x}_1 - \bar{x}_3) \pm t_{24}(\frac{0.05}{2 \times 2})\sqrt{\frac{54.506}{25}} = 1.88 \pm 2.3909 \times 1.4766 = [-1.6504, 5.4104]$$

$(\mu_2 - \mu_4)$:

$$(\bar{x}_2 - \bar{x}_4) \pm t_{24}(\frac{0.05}{2 \times 2})\sqrt{\frac{28.106}{25}} = 1.88 \pm 2.3909 \times 1.0603 = [-0.6551, 4.4151]$$

(d): 95% Scheffé type simultaneous C.I.:

$(\mu_1 - \mu_3)$:

$$(\bar{x}_1 - \bar{x}_3) \pm \sqrt{\frac{(n-1)p}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{54.506}{25}} = 1.88 \pm \sqrt{7.1374} \times 1.4766 = [-2.0649, 5.8249]$$

$(\mu_2 - \mu_4)$:

$$(\bar{x}_2 - \bar{x}_4) \pm \sqrt{\frac{(n-1)p}{n-p}F_{p,n-p}(0.05)}\sqrt{\frac{28.106}{25}} = 1.88 \pm \sqrt{7.1374} \times 1.0603 = [-0.9527, 4.7127]$$