THE UNIVERSITY OF CHICAGO

MODELING THE STOCK PRICE PROCESS AS A CONTINUOUS TIME JUMP PROCESS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF STATISTICS

BY
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CHICAGO, ILLINOIS
JUNE 2004

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Abstract

An important aspect of the stock price process, which has often been ignored in the financial literature, is that prices on organized exchanges are restricted to lie on a grid. We consider continuous-time models for the stock price process with random waiting times of jumps and discrete jump size. We consider a class of jump processes that are "close" to the Black-Scholes model in the sense that as the jump size goes to zero, the jump model converges to geometric Brownian motion. We study the changes in pricing and hedging caused by discretization. The convergence, estimation, discrete time approximation, and uniform integrability conditions for this model are studied. Upper and lower bounds on option prices are developed. We study the performance of the model with real data.

In general, jump models do not admit self-financing strategies for derivative securities. Birth-death processes have the virtue that they allow perfect hedging of derivative securities. The effect of stochastic volatility is studied in this setting. A Bayesian filtering technique is proposed as a tool for risk neutral valuation and hedging. This emphasizes the need for using statistical information for valuation of derivative securities, rather than relying on implied quantities.

Acknowledgments

I am most grateful to my thesis adviser Professor Per A. Mykland for his guidance, encouragement and enthusiasm. He introduced me to exciting research areas including this dissertation topic, provided me with his brilliant ideas and extremely valuable advice.

I am also very grateful to Professor Steven P. Lalley for his continuous encouragement and valuable advice. I would like to thank him for giving my first course in Mathematical Finance which got me interested in the field. My sincere thanks to Dr. Wei Bao for reviewing the first draft of this dissertation very carefully. It would have been impossible to complete this dissertation without the help of Dr. Kenneth Wilder. He provided me with all the data and indispensable guidance for the computations. I would also like to thank Professor George Constantinides for providing insight and valuable comments on the dissertation.

Finally I would like to express my gratitude to my husband Prabuddha Chakraborty who has been the greatest inspiration in my life and to my father Ashim K. Sen for showing me the way.

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Chapter 1

INTRODUCTION

Most of the standard literature in finance for pricing and hedging of contingent claims assumes that the underlying assets follow a geometric Brownian motion. Various empirical studies show that such models are inadequate, both in descriptive power, and for the mis-pricing of derivative securities that they might induce. Also the micro-structure predicted by these models includes observable quadratic variation (and hence volatilities), whereas this is nowhere nearly true in practice. The success and longevity of the Gaussian modeling approach depends on two main factors: firstly, the mathematical tractability of the model, and secondly, the fact that in many circumstances the model provides a simple approximation to the observed market behavior. A number of recent papers , including those by Hobson and Rogers(1998), Kallsen and Taqqu(1998), Melino and Turnbull(1990), Bates(1996), Bakshi et al. (1997) have stressed the importance of jump components and stochastic volatilities in option pricing. As an alternative, jump-diffusion models have been proposed, which are superpositions of jump and diffusion processes.

An alternative approach is to use pure-jump models. Eberlein and Jacod(1997) argue why a pure-jump process is more appropriate than a continuous one. The case for modeling asset price processes as purely discontinuous processes is also presented in a review paper by Madan(1999). The arguments address both the empirical realities of asset returns and the implications of the economic principle of no arbitrage. Some popular pure-jump models are the Variance Gamma model of Madan et al(1998), Normal Inverse Gaussian model of Barndorff-Neilsen(1998), Hyperbolic Distribution of Eberlein and Keller(1995), the CGMY process of Carr et al(2002). These are all parametric models and there is no clear way to verify the model assumptions. We take a distribution-free approach with minimal model assumptions and compute the range of values the option price can take over all possible jump distributions that belong to a large class. A somewhat similar approach is taken by Eberlein and Jacod(1997) who consider the class of all pure-jump Levy processes. However, the bounds that they derive for option prices are too large to be practicable.

Another aspect of the price process that has often been ignored is that security prices typically move in fixed units like 1/16 or 1/100 of a dollar. This does not present any particular problem when data are observed, say daily. Current technology however permits almost continuous observation, and estimation procedures based on discretely observed diffusions would then require throwing away data so as to fit the model. Harris(1991) and Brown et al(1991) argue the economic reason for the traders as well as institutions to maintain a non-trivial tick-size. Gottlieb and Kalay(1985) and Ball(1988) examine the biases resulting from the discreteness of observed stock prices. More references are given in section 2.4. However, there has been no attempt to integrate this discreteness with jump processes. All models involving discreteness are either discrete time or rounding of an underlying continuous path model at fixed points of time. In this paper we attempt to integrate the randomness of jump times with the discreteness of jump sizes.

As soon as we move out of the realm of continuous processes, the market becomes incomplete and the distribution of stock prices is not uniquely determined by no-arbitrage restrictions. We consider a class of jump processes that are "close" to the Black-Scholes model in the sense that as the jump size goes to zero, the jump model converges to geometric Brownian motion, which is the process for stock prices in the Black-Scholes model. We do not assume any further structure on the distributions. This requirement of convergence gives us the rate of events of the jump process and the first few moments of the jumps. Restricting to these models produces bounds on option prices that are small enough to be of practical use, without imposing further assumptions on the model. Thus we get an idea of how much difference it makes if we release the continuous path and normality assumptions of Brownian motion. We impose very few moment conditions, thereby allowing the thick tailed distributions that are observed in the empirical study of stock prices. The purpose of the paper is two-fold: first, to study the deviation of option prices from those predicted by continuous models; and second, to obtain the range of option prices when the distribution of the stock price belongs to the class of discontinuous models under consideration.

The very general jump models proposed in this paper do not render themselves to creating self financing strategies for derivative securities. This is a common phenomenon for general models with jumps and is the reason why continuous models are used. Here we come to a conflict: whereas from a data description point of view, it would make sense to use models with jumps, from a hedging standpoint, these models cannot be used. One of the consequences of this conflict is that statistical information is not used as much as it should be when it comes to valuing derivative securities. Instead there is a substantial reliance on "implied quantities", as in Beckers(1981), Engle and Mustafa(1992), Bick and Reisman(1993). It is shown in Mykland(1996) that this disregard for historical data can lead to mis-pricing. It would be desirable to bring as much statistical information as possible to bear on financial modeling and at the same time be able to hedge. This is why we think of birth-death models.

Birth-death processes have the virtue that they allow perfect derivative securities hedging. This is not quite as straightforward as the continuous model. In the latter, in simple cases, options need only be hedged in the underlying security; in birth-death process models one also needs one market traded derivative security to implement a self financing strategy. However, this is much nicer situation than for general models with jumps. In a sense birth death processes are almost continuous, as one needs to traverse all intermediate states to go from one point to the other. On the other hand, birth-death processes have a micro-structure which conflicts much less with the data.

The idea of modeling stock prices by a jump model in which they can go up, go down or stay the same was suggested in Perrakis(1988) to describe thinly traded stocks. The problem of pricing and hedging options in birth-death models where the rate is linear in the value of the stock is solved in Korn et al(1998). We show that a discretized version of geometric Brownian motion is obtained by considering a quadratic model. So we consider birth-death models where the rate is proportional to the square of the value of the stock. Also, in Korn et. al. (1998) the market has been completed with a very special option, the LEPO-put. We show that one can use any general option to complete the market. It should be noted that the methods developed here can be used as long as the intensity is of the form $\lambda_t g(S_t)$ and the drift is independent of the intensity.

The next step is the introduction of stochastic intensity. The intensity of jump processes is analogous to volatility in continuous models. Both theoretical and empirical considerations support the need for stochastic volatility. Asset returns have been modeled as continuous processes with stochastic volatility in Hull and White(1987), Naik(1993), Johnson and Shanno(1987), Heston(1993) or as jump processes with stochastic volatility in Bates(1996 and 2000), Duffie et al(2000).

The prior on the intensity process that we study in detail is a two state Poisson jump process. This assumes that stock price movements fluctuate between low and high intensity regimes. This is the approach in Naik(1993). We provide formulas for pricing and hedging options with any given prior on the intensity process. For example, alternatively we can consider cases where the intensity follows a diffusion as in Hull and White(1987), Wiggins(1987).

We obtain the risk neutral measure and the posterior under the risk neutral measure. What we are doing is an Empirical Bayes approach where the hyper-parameters are estimated from the data: the observed price of market traded options. We can test the goodness of fit of the model by comparing the implied intensity process to posterior. For the continuous models, we require extra assets for hedging when we introduce stochastic volatility. This is no more the case for birth-death models since the volatility is unobserved.

It is possible to generalize the birth and death to Poisson-type processes with finitely many jump amplitudes. The problem with this general jump model is that risk neutrality alone is not sufficient to uniquely determine the jump distribution. We have to either assume the jump distribution, or estimate it, or impose some optimization criterion (see e.g., Colwell and Elliot(1993), Elliot and Follmer(1991), Follmer and Schweizer(1990), Follmer and Sonderman(1986), Schweizer(1990 and 1993)). If the jump distribution is supported on n points, then we need $n-1$ market traded options to hedge a given option. This idea is taken up in Jones(1984).

The paper is organized as follows. In chapter 2 we discuss the historical background and related existing literature. In chapter 3 the general jump model is described and its convergence and estimation issues are studied . We provide the algorithm for obtaining bounds on option prices and compare these to the Eberlein and Jacod(1997) bounds in chapter 4.2.1. The birth death model is presented in chapter 5 . In chapters 6 and 7 we obtain the pricing and hedging strategies for birth and death models with constant and stochastic volatility respectively. We describe the Bayesian filtering techniques for updating the prior in chapter 8. In chapters 9 and 10 respectively, we present some simulation results and real-data applications.

Chapter 2

BACKGROUND AND EXISTING LITERATURE

2.1 Black-Scholes

In the Black and Scholes(1973) model, stock prices evolve according to a geometric Brownian motion. Despite its popularity this model has serious deficiencies; it provides inaccurate description of the distribution of returns and the behaviour of price-paths. If a model is based on the daily returns of a stock, statistical tests clearly reject the normality assumption made in the Black-Scholes case. For a more recent empirical study of distributions using German stock price data see Eberlein and Keller(1995). References to a number of classical studies in the US-market are given there. Looking at paths on an intra-day time-scale, that is looking at the micro-structure of stock price movements, Fig 3 of the same paper shows that a more realistic model should be a purely discontinuous model than a continuous one. Hansen and Westman(2002) analyze the path properties and find the existence of large jumps or extreme outliers. The distribution of returns are negatively skewed such that the left tail is thicker than the right tail.

Another drawback of the model concerns the anomaly of implied volatility. It is observed in Fortune (1996) that the implied volatility is an upwardly biased estimate of the observed volatility. The same paper notes 'volatility smile' in option markets. This refers to the general observation that near the money options tend to have lower implied volatilities than moderately in-the-money or out-of-the-money options. Also, puts tend to have higher implied volatilities than equivalent calls, indicating that puts are overpriced relative to calls. The overpricing is not random but systematic, suggesting that unexploited opportunities for arbitrage profits might exist.

According to continuous time models, the integrated volatility equals the quadratic variation. Hence, if data are observed continuously, the volatility should be observable. In practice we only observe a sample of the continuous time path. As shown in, for example, Jacod and Shiryaev (2002), the difference between the quadratic variation at discrete and continuous time scales converges to zero as the sampling interval goes to zero. Theorems 5.1 and 5.5 of Jacod and Protter(1998) and Proposition 1 of Mykland and Zhang(2001) give the size of the error in various cases. Hence, the best possible estimates of integrated volatility should be the observed quadratic variation computed from the highest frequency data obtainable. However, it has been found empirically that there is a bigger bias in the estimate when the sampling interval is quite small. Also, the estimate is not robust to changes in the sampling interval. Some references in this area include Brown(1990), Campbell et al(1997), Figlewski(1997), Andersen et al(2001). Thus, although the volatility should be asymptotically observable, this is not true in practice if data are available at very high frequency. One can still explain the phenomenon in the context of a continuous model (Ait-Sahalia, Mykland and Zhang (2003), Zhang, Mykland and Ait-Sahalia (2003)), but we have here chosen a different path.

2.2 Jump-diffusion

The jump diffusion model was introduced by Merton(1976). This model assumes that returns are IID. In particular, the returns usually behave as if drawn from a normal distribution but periodically "jumped" up or down by adding an independent normally distributed shock. The arrival of these jumps is random and their frequency is governed by the Poisson distribution with a given expected frequency. The advantage of the jump diffusion model is that it can make extreme events appear more frequently. The jumps are necessary to incorporate big crashes that are so frequently observed in the market. They are also more suitable in view of path properties of stock prices which are traded and recorded in integer multiples of 1/16 of a dollar. Aase (1984) as well as, with some restrictions, Eastham and Hastings (1988), Hastings (1992) have attempted to integrate jumps into portfolio selection. A comparative survey can be found in Duffie and Pan(1997). Other recent papers in this area are Kou(2002) which considers double exponential jump amplitudes and Hansen and Westman(2002) with log normal jumps. As noted in the introduction, there is no unanimous way to chose the jump distribution. Also, there is no justification to retain a continuous component other than mathematical simplicity and historical precedence.

2.3 Pure jump Processes

Eberlein and Jacod(1997) model the return process as a Levy process, that is a processes with stationary independent increments starting at 0, whose continuous martingale part vanishes. A typical example is the hyperbolic Levy motion defined in Eberlein and Keller(1995). For these incomplete models the no arbitrage approach alone does not suffice to value contingent claims. The class of equivalent martingale measures, which provides the candidates for risk neutral valuation, is by far too large. Additional optimality criteria or preference assumptions have to be imposed. Various attempts have been made to choose a particular probability. Follmer and Sondermann(1986) emphasize the hedging aspect and look for strategies that minimize the remaining risk in a sequential sense. Follmer and Schweizer(1990) study a minimal martingale measure in the sense that it minimizes relative entropy. Another approach is variance optimality. This means to choose the martingale measure whose density is minimized in the $\mathcal{L}^2$-sense as in Schweizer(1996). Also the Esscher transform used by Eberlein and Keller(1995) to derive explicit option values seems to be a natural choice (eg. generalized hyperbolic model and CGMY model). Other papers in this area are Geman(2002), Eberlein(2001), Konikov and Madan (2002), Bingham and Kiesel(2001).

2.4 Discreteness

Gottlieb and Kalay(1985) observe that stock prices on organized exchanges were restricted to be divisible by 1/8. This paper examines the biases resulting from the discreteness of observed stock prices. Modeling true price $P(t)$ as log-normal diffusion and the observed price $\hat{P}(t)$ as point on the grid closest to the true price, it is shown that the natural estimator of the variance and all of the higher moments of the rate of returns are biased. Ball (1988) examines the probabilistic structure of the resultant rounded process, provides estimates of inflation in estimated variance and kurtosis induced by ignoring rounding. Harris(1990) shows that the discreteness increases return variance and adds negative serial correlation to return series. Anshuman and Kalay (1998) compute the economic profits arising from discreteness. Brown et al. (1991) argues that traders endogenously choose a tick size to control bargaining costs. Grossman and Miller(1988) suggest that the minimum tick size ensures profits on quick turnaround transactions. Cho and Frees(1988) discuss the estimation of volatility under the model introduced by Gottlieb and Kalay(1985). Other papers in this area include Harris(1991), Harris and Lawrence(1997), Hasbrouk(1999a & b).

2.5 Neural Networks

There have been some attempts to price and hedge options using this is a distribution free fitting algorithm as in Anders et al(1998). They are becoming more popular with increase in computational power. They do not provide any theoretical results or economic insights into the dynamics of the system. Empirical studies show that for volatile markets a neural network option pricing model outperforms the traditional Black-Scholes model. However the Black-Scholes model is still good for pricing at-the-money options, see Yao et al(2000). By testing for the explanatory power of several variables serving as network inputs, some insight into the pricing process of the option market is obtained.

Chapter 3

THE PROPOSED MODEL

We start with the simple model of jumps of size $\pm c$ and event rate proportional to the present stock price (linear jump rate). This is the discrete state-space version of the popular affine jump diffusion models, for example see Duffie et al(1999). This model is also studied by Korn et al(1998) where, assuming that the risk-neutral distribution is a linear jump process, they obtain the implied jump rates by inverting the price of a market traded option and price other options using these rates. However, we show that the linear intensity birth-death model with constant intensity rate is not adequate. In section 3.1, we describe the linear birth-death model of Korn et al(1998) and show that the stock price process under this model converges in probability as $c \to 0$ to a deterministic process. So we need to either change the event rate or introduce jumps of size greater than 1. In Chapters 5 to 8, we study the quadratic (intensity proportional to square of stock price) birth-death model with random event rate. In section 3.2 we introduce jumps of size bigger than one and for the rest of chapters 3 and 4.2.1 we study these general jump models with jump size greater than 1 and linear intensity with constant rate. We state the precise theorem and conditions involving convergence of the jump models to geometric Brownian motion, the continuous path models for stock prices in Black-Scholes option pricing theory. This convergence result is a general technique. In fact, if the underlying security is believed to have different properties than those predicted by the Black-Scholes model in the limit, then we can similarly derive different conditions on the class of "close" jump processes. As an illustration, similar conditions for convergence to the Cox-Ingersoll-Ross model are stated. Hence under these modified conditions, we have a model for interest rates that is the discretized version of the Cox-Ingersoll-Ross model. In section 3.3, we discuss the estimation of parameters from historical stock price data for models that satisfy the conditions stated in section 3.2. Finally, in section 3.4 we describe the method of introducing rare big jumps.

3.1 The linear birth-death model

Suppose that the stock price S_t is a birth and death process with jump size c, jump intensity $\lambda_t S_t/c$, and probability of a positive jump p_t for some positive parameters λ_t and p_t. Let $N_t = S_t/c$. For example, S_t is price of stock in dollars, N_t in cents, $c = 1/100$. N_t is modeled as a *non-homogeneous* (birth and death rates per individual depend on t), *linear* (rates are proportional to number of individuals present) birth and death process. We suppose that there is a risk-free interest rate ρ_t. Let $\mathrm{P}_k(t) = \mathrm{P}(N_t = k)$. The price of an option with payoff $f(S_T)$ is $\mathrm{E}[\exp\{-\int_0^T \rho_s ds\} f(S_T)] = \exp\{-\int_0^T \rho_s ds\} \sum_{k=0}^{\infty} \mathrm{P}_k(T) f(ck)$. The Kolmogorov's forward equations are:

$$\begin{aligned}
\mathrm{P}'_k(t) &= -k\lambda_t \mathrm{P}_k(t) + (k-1)\lambda_t p_t \mathrm{P}_{k-1}(t) + (k+1)\lambda_t(1-p_t)\mathrm{P}_{k+1}(t) \quad k \geq 1 \\
\mathrm{P}'_0(t) &= \lambda_t(1-p_t)\mathrm{P}_1(t) \quad k \geq 1
\end{aligned}$$

$$\phi(u,t) = \sum_{k=0}^{\infty} \mathrm{P}_k(t) u^k = \left[1 - \frac{1}{\frac{1}{a_t(1-u)} + b_t}\right]^{N_0} \tag{3.1}$$

where $a_t = \exp\{\int_0^t \lambda_s(2p_s - 1)\, ds\}$ and $b_t = \int_0^t \lambda_s p_s / a_s ds$. The derivation of $\phi(u,t)$ is given in Appendix D. We can obtain $\mathrm{P}_k(t)$ as the coefficient of u^k in the power series expansion of $\phi(u,t)$.

$$\begin{aligned} \mathrm{P}_k(t) &= \sum_{j=0}^{k} \frac{(N_0+j-1)!N_0}{j!(k-j)!(N_0-k+j)!} \left(\frac{b_t}{a_t^{-1}+b_t}\right)^j \\ &\quad \times \left(\frac{1-b_t}{a_t^{-1}+b_t-1}\right)^{k-j} \left(\frac{a_t^{-1}+b_t-1}{a_t^{-1}+b_t}\right)^{N_0} \end{aligned}$$

$$\mathrm{E}(N_t) = \frac{\partial \phi}{\partial u}\bigg|_{u=1} = N_0 \left(1 - \frac{1}{\frac{1}{a_t(1-u)}+b_t}\right)^{N_0-1} \frac{\frac{1}{a_t(1-u)^2}}{\left(\frac{1}{a_t(1-u)}+b_t\right)^2}\Bigg|_{u=1} = N_0 a_t$$

Similarly, we can derive the variance of N_t. The distribution of N_t is the sum of N_0 iid random variables with mean a_t and variance $(a_t^{-1}+2b_t-1)a_t^2$. So $\mathrm{E}(S_t) = S_0 a_t$ and $\mathrm{Var}(S_t) = c(a_t^{-1}+2b_t-1)a_t^2 S_0$. Due to arbitrage requirements, $\exp\{-\int_0^t \rho_u du\}S_t$ is a martingale. This specifies p_t uniquely as $\frac{1}{2}(1+\frac{\rho_t}{\lambda_t})$ and $a_t = \exp\{\int_0^t \rho_u du\}$.

$$\mathrm{P}(\mid S_t - e^{\int_0^t \rho_s ds} S_0 \mid > \epsilon) \leq \frac{c(a_t^{-1}+2b_t-1)a_t^2 S_0}{\epsilon^2}$$

When $c \longrightarrow 0$, $S_t \xrightarrow{\mathrm{P}} \exp\{\int_0^t \rho_s ds\}S_0$. Thus the simple model of jumps of size $\pm c$ and event rate proportional to the present stock price converges to a deterministic process in probability.

3.2 Introducing distribution on the size of jumps

Suppose now that for each n, the stock price $S_t^{(n)} = N_t^{(n)}/n$ where $N_t^{(n)}$ is a sequence of integer valued jump processes. That is, the grid size is $c = 1/n$ and we consider a sequence of random processes with grid size decreasing to 0. We assume initial stock price $S_0^{(n)}$ is the same for all n. The number of jumps $\xi_t^{(n)}$ is assumed to be a counting process with rate $N_t^{(n)}\sigma_t^2$ and the random jump size of $N_t^{(n)}$ is denoted by $Y_t^{(n)}$. Let $\mathcal{F}_u^{(n)} = \sigma\{N_u, 0 \leq u \leq t\}$. Under some assumptions on the conditional distribution of $Y_t^{(n)}$ that are outlined in Proposition 3.2.0.1, as $n \to \infty$, the sequence of random processes $S_t^{(n)}$ converge in distribution to process S_t which evolves as

$$\ln S_t = \ln S_0 + \int_0^t (\rho_u - \frac{1}{2}\sigma_u^2)du + \int_0^t \sigma_u dW_u \tag{3.2}$$

where W_t is standard Weiner process. This is the stochastic differential equation governing the stock price process in the Black-Scholes model of asset pricing.

To illustrate that this method is quite general, we can consider the interest rate process. Suppose the interest rate process $R_t^{(n)} = N_t^{(n)}/n$ where the process $N_t^{(n)}$ is as described above. We assume initial interest rate $R_0^{(n)}$ is the same for all n. Under some assumptions on the conditional distribution of $Y_t^{(n)}$ that are outlined in Proposition 3.2.0.2, as $n \to \infty$, the sequence of random processes $R_t^{(n)}$ converge in distribution to process R_t which evolves as :

$$R_t = R_0 + \int_0^t a(b - R_u)du + \int_0^t \sigma\sqrt{R_u}dW_u \tag{3.3}$$

where W_t is standard Weiner process. This is the stochastic differential equation governing the interest rate according to the Cox-Ingersoll and Ross model for interest rates (Ref section 21.5 of Hull(1999))

Proposition 3.2.0.1. *Let $N_t^{(n)}$ and $Y_t^{(n)}$ be as described above. Then the process $S_t^{(n)} = N_t^{(n)}/n$ converges in distribution to a process S_t which evolves as in (3.2) if:*

$$\sup_{s\leq t}\left|\ln\left(1 + \frac{Y_s^{(n)}}{N_{s-}^{(n)}}\right)\right| \xrightarrow{\text{P}} 0 \quad \forall t \tag{B1}$$

$$\int_0^T \mathrm{E}\left[\ln\left(1+\frac{Y_t^{(n)}}{N_{t-}^{(n)}}\right)\Big|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\sigma_t^2 dt \xrightarrow{\mathrm{P}} \int_0^T (\rho_t - \sigma_t^2)dt \tag{B2}$$

$$\int_0^T \mathrm{E}\left[\left\{\ln(1+\frac{Y_t^{(n)}}{N_{t-}^{(n)}})\right\}^2 \Big|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\sigma_t^2 dt \xrightarrow{\mathrm{P}} \int_0^T \sigma_t^2 dt \tag{B3}$$

A set of sufficient conditions for (B2)-(B3) to hold is: $\mathrm{E}[Y_t^{(n)} \mid \mathcal{F}_{t-}^{(n)}] = \rho_t/\sigma_t^2$, $\mathrm{E}[Y_t^{(n)2} \mid \mathcal{F}_{t-}^{(n)}] = N_{t-}^{(n)}$ and $\mid Y_t^{(n)} \mid \leq kN_{t-}^{(n)\delta}$ where $0 < k < 1$ and $\delta < 2/3$.

Proof. The proof is given in Appendix A. □

Proposition 3.2.0.2. *Let $N_t^{(n)}$ and $Y_t^{(n)}$ be as described above. Then the process $R_t^{(n)} = N_t^{(n)}/n$ converges in distribution to a process R_t, which evolves as in (3.3), if*

$$\sup_{s\leq t}\left|\sqrt{Y_s^{(n)} + N_{s-}^{(n)}} - \sqrt{N_{s-}^{(n)}}\right| \xrightarrow{\mathrm{P}} 0 \quad \forall t \tag{C1}$$

$$\int_0^T \left\{\mathrm{E}\left[\left(\sqrt{Y_t^{(n)} + N_{t-}^{(n)}} - \sqrt{N_{t-}^{(n)}}\right)\Big|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\frac{\sigma_t^2}{\sqrt{n}} - \frac{a(b-N_{t-}^{(n)}/n)}{\sqrt{N_{t-}^{(n)}/n}}\right\} dt \xrightarrow{\mathrm{P}} 0 \tag{C2}$$

$$\int_0^T \mathrm{E}\left[\left(\sqrt{Y_t^{(n)} + N_{t-}^{(n)}} - \sqrt{N_{t-}^{(n)}}\right)^2 \Big|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\frac{\sigma_t^2}{n} dt \xrightarrow{\mathrm{P}} \int_0^T \sigma_t^2 dt \tag{C3}$$

A set of sufficient conditions for (C2)-(C3) to hold is: $\mathrm{E}[Y_t^{(n)} \mid \mathcal{F}_{t-}^{(n)}] = ab/(N_{t-}^{(n)}\sigma^2) - (a-\sigma^2/4)/(n\sigma^2)$, $\mathrm{E}[Y_t^{(n)2} \mid \mathcal{F}_{t-}^{(n)}] = n$ and $\mid Y_t^{(n)} \mid \leq kN_{t-}^{(n)\delta}$ where $0 < k < 1$ and $\delta < 2/3$. According to Feller(1951), initial values can be prescribed arbitrarily for the model (3.3) and they uniquely determine a solution. This solution is positivity preserving and norm decreasing.

Proof. The proof is similar to that of Proposition 3.2.0.1 □

For the rest of chapters 3 and 4, unless otherwise mentioned, we shall restrict ourselves to stock price processes S_t, that under the physical measure are pure-jump processes with the jump time ξ_t, a counting process with rate $S_t/c\lambda$ and jump size cY_t, where Y_t is an integer valued random variable with $\mathrm{E}(Y_t \mid S_t) = \nu$ and $\mathrm{E}(Y_t^2 \mid S_t) = S_t/c$. We shall denote the class of probability measures associated with such processes by $\mathcal{M}$. We have shown that if we let c to go to zero, then under some regularity conditions, such processes converge to geometric Brownian motion with drift λ/ν and volatility λ. We shall denote $\mathrm{P}(Y_t = i \mid S_t = cj)$ by $p(i,j)$.

3.3 Estimation

For the processes under consideration, we have two unknown parameters λ and ν. For continuous models the quadratic variation is predictable, and in the no-arbitrage setting, we can invert some option prices to get the implied volatility under the risk-neutral measure. However, in the jump model setting this is no longer true and the volatility needs to be estimated from observed data. We can do an inversion to get an implied λ here, too. But there is no theory to justify that it is the volatility under the risk-neutral measure, because the market is incomplete and the risk-neutral measure need not be unique. From a statistical point of view, the implied volatility is a method of moments estimator and we do not know what optimality properties it has. Here we propose an alternative quasi-likelihood estimator and study its properties.

Suppose we observe the process from time 0 to τ and the jumps occur at times $\tau_1, \tau_2 \ldots \tau_k$. Let $\tau_0 := 0$. Conditional on $(S_{\tau_1}, S_{\tau_2}, \ldots, S_{\tau_k})$, $\tau_i - \tau_{i-1}$ for $i = 1, \ldots, k$ are independent exponential random variables with parameter $S_{i-1}\lambda/c$. The non-parametric maximum likelihood estimator (Ref. IV.4.1.5 of Andersen et al(1993)) is obtained by maximizing

$$\prod_{i=1}^{k} \frac{S_{\tau_{i-1}}}{c} \lambda^{2k} p\left(\frac{\Delta S_{\tau_i}}{c}, \frac{S_{\tau_{i-1}}}{c}\right) \exp\left\{-\left[\sum_{i=1}^{k} \frac{S_{\tau_{i-1}}}{c}(\tau_i - \tau_{i-1}) + \frac{S_{\tau_k}}{c}(T - \tau_k)\right]\lambda^2\right\}$$

$$\text{The NPMLE is:} \qquad \hat{\lambda} = k\left[\sum_{i=1}^{k} \frac{S_{i-1}}{c}(\tau_i - \tau_{i-1}) + \frac{S_k}{c}(T - \tau_k)\right]^{-1}$$

Since the distribution of the jump size is not uniquely determined by convergence conditions, we do not have the likelihood of ν.

$$\mathrm{E}(S_{\tau_i} \mid S_{\tau_{i-1}} = x) = x + \nu = m_\nu(x)$$

$$\mathrm{E}((S_{\tau_i} - x)^2 \mid S_{\tau_{i-1}} = x) = \mathrm{E}((cY_{\tau_i})^2 \mid S_{\tau_{i-1}} = x) = cx$$

$$\mathrm{Var}(S_{\tau_i} \mid S_{\tau_{i-1}} = x) = cx - (x + \nu)^2 = v_\nu(x)$$

As shown in Wefelmeyer(1996), a large class of estimators for ν is obtained as solutions of estimating equations of the form

$$\sum_{i=1}^{n} w_\nu(S_{\tau_{i-1}})(S_{\tau_i} - m_\nu(S_{\tau_{i-1}})) = 0$$

Under appropriate conditions the corresponding estimator is asymptotically normal. The variance of this estimator is $\pi(w_\nu^2 v_\nu)/(\pi(w_\nu m_\nu'))^2$. Here $\pi(f)$is short for the expectation $\int f(x)\pi(dx)$, and prime denotes differentiation with respect to ν. By Schwartz inequality, the variance is minimized for $w_\nu = m_\nu'/v_\nu$. In our case, the estimating equation for the estimator with minimum variance becomes the solution of

$$\sum_{i=1}^{n} \frac{S_{\tau_i} - S_{\tau_{i-1}} - \nu}{S_{\tau_{i-1}}\nu - (S_{\tau_{i-1}} + \nu)^2} = 0$$

3.4 Introducing rare big jumps

It is easy to generalize this model to include rare big jumps. Assume that the jumps come from two competing processes.

- Small jumps with rate $N_t\lambda_t$, $\mathrm{E}(Y_t) = \frac{r_t}{\lambda_t}$, $\mathrm{E}(Y_t^2) = N_t$
- Big jumps with rate μ_t, $\mathrm{E}(Y_t) = aN_t$, $\mathrm{E}(Y_t^2) = bN_t^2$

In the limit this converges to the Merton Jump-diffusion model. Estimation can be done by EM algorithm regarding the data augmented with the source of jump as complete data.

Chapter 4

UPPER AND LOWER BOUNDS ON OPTION PRICE

We shall assume, for this chapter, that there is a constant interest rate ρ. We restrict ourselves to the class $\mathcal{M}$ of probability measures described in the end of section 3.2 with $\nu = \rho/\lambda$. We estimate the parameters as shown in section 3.3. Even then, we do not have a unique distribution for the stock price. This is because the distribution of jump size is not uniquely specified by the conditions imposed. We get a class of models each of which gives a different price for options. A similar problem is addressed in Eberlein and Jacod (1997) who consider the class of all pure jump Levy processes. They derive upper and lower bounds for option prices when the distribution of the stock price process belongs to a large class of distributions. In section 4.1 we present the Eberlein-Jacod bounds and show that these bounds hold if the class of distributions is $\mathcal{M}$. We show that in this case, there exists a smaller upper bound than the Eberlein and Jacod upper bound. We also show that the lower bound is sharp; that is, there exist a sequence of distributions in $\mathcal{M}$, under which the option price converges to the Eberlein and Jacod lower bound. We cannot obtain any sharp upper bounds theoretically for the models under consideration. So in section 4.2 we present an algorithm to obtain these. This algorithm is very computationally intensive. As an alternative, in section 4.3 we derive an algorithm for obtaining the bounds for a discrete time approximation to the stock price process.

4.1 Comparing to Eberlein-Jacod bounds

Suppose there is a constant interest rate ρ. Let $\gamma(Q) = \mathrm{E}_Q[e^{-\rho T} f(S_T)]$ be the expected discounted payoff of an option under the measure Q. Assume

$$f \text{ is convex, and} \quad 0 \leq f(x) \leq x \quad \forall x > 0 \tag{D}$$

Lemma 4.1.0.1. *Under each $Q \in \mathcal{M}$, $e^{-\rho t} S_t$ is a martingale.*

Proof. The proof is given in Appendix D ☐

It is shown in Eberlein and Jacod(1997), that under reasonable conditions on Q, and f satisfying (D), the following holds:

$$e^{-\rho T} f(e^{\rho T} S_0) < \gamma(Q) < S_0 \tag{4.1}$$

Proposition 4.1.0.3. *The Eberlein-Jacod bounds 4.1 hold for $Q \in \mathcal{M}$ and f satisfying (D).*

Proof. Since f is convex, by Lemma 4.1.0.1, under each $Q \in \mathcal{M}$ the process $A_t = f(e^{\rho(T-t)} S_t)$ is a Q-submartingale. So $\gamma(Q) = e^{-\rho T} \mathrm{E}_Q[A_T] \geq e^{-\rho T} f(e^{\rho T} S_0)$. We have $e^{-\rho T} f(S_T) < e^{-\rho T} S_T$ by assumption (D). So $\gamma(Q) < \mathrm{E}_Q[e^{-\rho T} S_T] = S_0$ ☐

Proposition 4.1.0.4. *There exists a smaller upper bound for $\gamma(Q)$ than that given in 4.1 when $Q \in \mathcal{M}$.*

Proof. Let ξ_t be the counting process of the number of jumps in the stock price.

$$\begin{aligned}
e^{\rho T}\gamma(Q) &= \mathrm{E}_Q[f(S_T)] \\
&= \mathrm{E}_Q[f(S_T)I_{\{\xi_T=0\}}] + \mathrm{E}_Q[f(S_T)I_{\{\xi_T>0\}}] \\
&\leq \mathrm{P}(\xi_T = 0)f(S_0) + \mathrm{E}_Q[S_T I_{\{\xi_T>0\}}] \\
&= \mathrm{P}(\xi_T = 0)f(S_0) + \mathrm{E}_Q[S_T] - E_Q[S_T I_{\{\xi_T=0\}}] \\
&= \mathrm{P}(\xi_T = 0)f(S_0) + e^{\rho T}S_0 - S_0\mathrm{P}(\xi_T = 0) \\
&= e^{\rho T}S_0 - (S_0 - f(S_0))\exp\{-\frac{S_0}{c}\lambda_0 T\}
\end{aligned}$$

$S_0 - \gamma(Q) = e^{-\rho T}(S_0 - f(S_0))\exp\{-\frac{S_0}{c}\lambda_0 T\} > 0$ □

Proposition 4.1.0.5. *Assuming $\rho = 0$, there exist a sequence of distributions $Q^{(m)} \in \mathcal{M}$ such that $\gamma(Q^{(m)})$ converges to the lower bound in 4.1 as $m \to \infty$*

Proof. Define the measure $Q^{(m)}$ as follows: For each value j of ξ_T and each m, define the Markov chain $S_k^{(m,j)}$ for $1 \leq k \leq j$ by

$$S_k^{(m,j)} = \begin{cases} S_{k-1}^{(m,j)} - \sqrt{S_{k-1}^{(m,j)}}\frac{\sqrt{c}}{\sqrt{m-1}} & w.p. \quad 1 - \frac{1}{m} \\ S_{k-1}^{(m,j)} + \sqrt{S_{k-1}^{(m,j)}}\sqrt{c}\sqrt{m-1} & w.p. \quad \frac{1}{m} \end{cases}$$

$\mathrm{E}[S_k^{(m,j)} - S_{k-1}^{(m,j)} \mid S_{k-1}^{(m,j)}] = 0$
$\mathrm{E}[(S_k^{(m,j)} - S_{k-1}^{(m,j)})^2 \mid S_{k-1}^{(m,j)}] = cS_{k-1}^{(m,j)}$
Hence $Q^{(m)} \in \mathcal{M}$

Claim 4.1.0.1. *Given δ and ϵ, for each j, we can get m_j such that for all $m > m_j$,*

$$\mathrm{P}(|S_T^{(m,j)} - S_0| > \delta \mid \xi_T = j) < \epsilon$$

Claim 4.1.0.2. *There exists J such that* $\mathrm{P}(\xi_T > J) < \epsilon$

Let $n = max_{j=1}^{J} m_j$. Then $\forall m > n, \forall j < J \quad \mathrm{P}(|S_T^{(m,j)} - S_0| > \delta \mid \xi_T = j) < \epsilon$

$$\begin{aligned} \mathrm{P}(|S_T^{(m)} - S_0| > \delta) &= \sum_{j=1}^{J} \mathrm{P}(|S_T^{(m)} - S_0| > \delta \mid \xi_T = j)\mathrm{P}(\xi_T = j) \\ &\quad + \mathrm{P}(|S_T^{(m)} - S_0| > \delta \mid \xi_T > J)\mathrm{P}(\xi_T > J) \\ &\leq \epsilon \times 1 + 1 \times \epsilon \\ &= 2\epsilon \end{aligned}$$

Hence $S_T^{(m)} - S_0 \xrightarrow{\mathrm{P}} 0$. $S_T^{(m)}$ is non-negative and $\mathrm{E}(S_T^{(m)}) = \mathrm{E}(S_0)$. So $\{S_T^{(m)}\}$ is uniformly integrable. This and assumption D implies $\{f(S_T^{(m)})\}$ is uniformly integrable. $S_T^{(m)} \xrightarrow{\mathrm{P}} S_0$ and f is continuous. So $f(S_T^{(m)}) \xrightarrow{\mathrm{P}} f(S_0)$. This and uniform integrability of $f(S_T^{(m)})$ implies $\mathrm{E}(f(S_T^{(m)})) \longrightarrow \mathrm{E}(f(S_0)) = f(S_0)$ □

Proof of **Claim 4.1.0.1**

$$\begin{aligned} &\mathrm{P}(| S_T^{(m,\xi_T)} - S_0^{(m,\xi_T)} | \leq \xi_T \sqrt{S_0^{(m,\xi_T)}}\sqrt{c}\frac{1}{\sqrt{m-1}} \mid \xi_T = j) \\ \geq\ & \mathrm{P}(S_j^{(m,\xi_T)} - S_{j-1}^{(m,\xi_T)} = -\sqrt{S_{j-1}^{(m,\xi_T)}}\frac{\sqrt{c}}{\sqrt{m-1}} \\ & \&\ldots \ \& S_1^{(m,\xi_T)} - S_0^{(m,\xi_T)} = -\sqrt{S_0^{(m,\xi_T)}}\frac{\sqrt{c}}{\sqrt{m-1}}) \\ =\ & (1 - \frac{1}{m})^j \qquad \square \end{aligned}$$

Proof of **Claim 4.1.0.2**

$$\mathrm{P}(\xi_T > J) \leq \frac{1}{J}\mathrm{E}(\xi_T) = \frac{1}{J}\mathrm{E}\int S_t \frac{\lambda}{c} dt = \frac{S_0 \lambda t}{cJ} < \epsilon \text{ for } J \text{ sufficiently large} \quad \square$$

When $\rho \neq 0$, we need to let the grid size go to 0 to obtain a sequence of measures that converge to the lower bound.

Proposition 4.1.0.6. *When $\rho \neq 0$, there exist a sequence of distributions $Q^{(c,m)} \in \mathcal{M}$ such that $\gamma(Q^{(c,m)})$ converges to $e^{-\rho T} f(S_0(1+\rho T))$ as $c \to 0$ and $m \to \infty$*

Proof. The proof of proposition 4.1.0.5 can be extended to nonzero interest rate with the following modifications: For each grid size c, define the Markov chain $S_k^{(m,\xi_T)}$ for $1 \leq k \leq \xi_T$ by

$$S_k^{(m,\xi_T)} = \begin{cases} S_{k-1}^{(m,\xi_T)} + \frac{c\rho}{\lambda} - \sqrt{S_{k-1}^{(m,\xi_T)} - \frac{c\rho^2}{\lambda^2}} \frac{\sqrt{c}}{\sqrt{m-1}} & w.p. \quad 1 - \frac{1}{m} \\ S_{k-1}^{(m,\xi_T)} + \frac{c\rho_t}{\lambda} + \sqrt{S_{k-1}^{(m,\xi_T)} - \frac{c\rho^2}{\lambda^2}} \sqrt{c}\sqrt{m-1} & w.p. \quad \frac{1}{m} \end{cases}$$

Given $\xi_T = j$ and all jumps are negative,

$$\begin{aligned} |S_T - S_0 - \frac{jc\rho}{\lambda}| &\leq \frac{\sqrt{c}}{\sqrt{m-1}} \sum_{i=1}^{j} \sqrt{S_i - \frac{c\rho^2}{\lambda^2}} \\ &\leq \frac{\sqrt{c}}{\sqrt{m-1}} \xi_T \sqrt{S_0 + \frac{jc\rho}{\lambda} - \frac{c\rho^2}{\lambda^2}} \overset{m\to\infty}{\longrightarrow} 0 \end{aligned}$$

So given δ, ϵ, j, can find m large enough so that $\mathrm{P}(|S_T^{(m,j)} - S_0 - \xi_T c\rho/\lambda| > \delta/2 \mid \xi_T = j) < \epsilon$. $\mathrm{P}(|S_T - S_0 - S_0\rho t| > \delta \mid \xi_T = j) = \mathrm{P}(|S_T - S_0 - \xi_T c\rho/\lambda| > \delta/2 \mid \xi_T = j) + \mathrm{P}(|\xi_T c\rho/\lambda - S_0\rho t| > \delta \mid \xi_T = j) < \epsilon + \epsilon$. This is because $c\xi_T \overset{\mathrm{P}}{\longrightarrow} \lambda S_0 T$ as $c \to 0$ since $d < \xi >_t = S_t\lambda/c$ and $d < \xi, \xi >_t = S_t\lambda/c$. The rest of the proof is same as the case $\rho = 0$ except that instead of S_0 we now have $S_0 + S_0\rho t$. $\mathrm{E}(f(S_T^m)) \longrightarrow \mathrm{E}(f(S_0 + S_0\rho t)) = f(S_0 + S_0\rho t)$. If ρt is small, $S_0 + S_0\rho t$ is approximately $S_0 e^{\rho t}$. $\gamma(Q_{m,c,\rho}) = \mathrm{E}(e^{-\rho t} f(S_T^{(m,c,\rho)})) \longrightarrow e^{-\rho t} f(S_0 e^{\rho t})$ as $m \to \infty, c \to 0, \rho \to 0$ □

We have seen in section 3.2 the conditions under which the jump process converges to geometric Brownian motion in law. Let $S^{(n)}$ be the jump process with grid size $c = 1/n$ and $S^{(n)} \stackrel{d}{\Longrightarrow} S$ where S is geometric Brownian motion. For any continuous function $f(x)$, $f(S_T^{(n)}) \stackrel{d}{\Longrightarrow} f(S_T)$. Hence any payoff under the jump process that is a continuous function of the stock price, in particular put and call options, converges in distribution to the payoff under the Black-Scholes model. We are interested in the price of options. For example we want to have $\mathrm{E}(S_T^{(n)} - K)_+ \longrightarrow \mathrm{E}(S_T - K)_+$. This will hold if we have uniform integrability of $(S_T^{(n)} - K)_+$. It can be shown that uniform integrability holds under Lyapounov condition (E). In this section we observe examples of jump processes under which the prices of options are very different from the Black-Scholes price. Note that these distributions do not satisfy (E).

$$c_n := E \sum_{0 \le t \le T} (\frac{Y_t}{n})^4 \stackrel{n\to\infty}{\longrightarrow} 0. \tag{E}$$

4.2 Obtaining the upper and lower bounds

In section 4.2.1 we describe a dynamic programming algorithm to get the maximum price of an option with payoff $f(S_T)$ when the distribution of the stock price process S_t belongs to the class $\mathcal{M}$. The same algorithm with the maximum at the intermediate steps replaced by minimum will give the minimum price. This procedure gives a range for possible option prices when the stock price process has a distribution $Q \in \mathcal{M}$. Let ξ_t=number of jumps in the stock price till time t and let $N_t = S_t/c$. The frequency distribution of ξ_T is obtained in Section 4.2.2

4.2.1 The Algorithm

- For each m such that $\mathrm{P}(\xi_T = m) > \epsilon$,

- For each $i \leq m$, going down over the integers

- For each value k of $N_{T_{i-1}}$

- maximize $\mathrm{E}(f_i(l + Y_i)|\xi_T = m, N_{T_{i-1}} = k)$ over the distribution on Y_i where

$$\begin{aligned} f_i(x) &= (x - \frac{K}{c})_+ && \text{for } i = m \\ f_i(x) &= \max \mathrm{E}(f(x + Y_i)|\xi_T = m, N_{T_{i-1}} = x) && \text{for } 1 \leq i \leq m - 1 \end{aligned}$$

The maximum value is $f_1(N_0)$. The problem reduces to maximizing $\mathrm{E}(f_i(l+Y_i)|\xi_T = m, N_{T_{i-1}} = k)$ over the distribution on Y_i. Let $p_{y,k} = \mathrm{P}(Y_i = y|N_{T_{i-1}} = k)$. We have the constraints: $\sum p_{y,k} = 1, \sum y p_{y,k} = \rho/\lambda, \sum y^2 p_{y,k} = k$

$$\mathrm{E}(f_i(l + Y_i)|\xi_T = m, N_{T_{i-1}} = k) = \frac{\sum_y f_i(l + Y_i) p_{y,k} \mathrm{P}(\xi_T = m|N_{T_{i-1}} = k, Y_i = y)}{\sum_y p_{y,k} \mathrm{P}(\xi_T = m|N_{T_{i-1}} = k, Y_i = y)}$$

$$\mathrm{P}(\xi_T = m|N_{T_{i-1}} = k, Y_i = y) = \int_0^T q_{i,k,y}(t) Q_{m-i,k+y}(T - t) dt$$

where $Q_{m,k}(t) = \mathrm{P}(\xi_t = m|N_0 = k)$ is given by Claim 4.2.2.1 and $q_{i,k,y}(t)$ is the conditional density of T_i given $N_{T_{i-1}} = k, Y_i = y$. To obtain $q_{i,k,y}(t)$, observe that $T_i = T_{i-1} + \Delta T_i$. The conditional distribution of ΔT_i is $\mathrm{Exp}(N_{i-1}\sigma^2)$. T_{i-1} is independent of $N_{T_{i-1}} = k, Y_i = y$. The unconditional distribution is: $\mathrm{P}(T_{i-1} \leq t) = \mathrm{P}(\xi_t \geq i - 1) = 1 - \sum_{j=0}^{l-2} Q_{j,N_0}(t)$

We have to maximize the ratio of 2 linear functions of p_y under three linear constraints. That is: max $x'y/x'z$ under three linear constraints on x. Suppose at the maximum $x'z = \mu$. Then at that point $x'y$ is maximized subject to 4 linear constraints. This will be a 4-point distribution. So the maximizing p_y is supported on 4 points. Let the four points be y_1, y_2, y_3, y_4. From the three constraints, we can express $p_{y_1}, p_{y_2}, p_{y_3}$ as linear functions of p_{y_4}. Then we have to maximize a ratio of 2 linear functions of p_{y_4}. This is a monotone function of p_{y_4}. Hence the maximum occurs at a boundary. So we actually have a three point distribution where the maximum is attained. The algorithm is to check through all the three point distributions of Y_i and check where the maximum occurs.

An alternative procedure here is to do linear programming. But it was found that both linear programming and checking through all possible three point distributions took comparable amount of computational time. In fact we can characterize and eliminate a lot of 3-point combinations from the search list since all p_is need to be positive and not all combinations satisfy this. On the other hand, since linear programming gives a numerical maximum, the result has the same minimum value numerically but is not in general supported on three points and is therefore difficult to interpret. Hence our simulations were all carried out by checking through all admissible three point distributions.

If the number of possible values of the stock price is n, then the number of possible jump combinations that we need to check naively is n^3. However, this number is greatly reduced since all these combinations cannot support probability distributions with the given constraints. Suppose the jump distribution is supported on three points $y_1 < y_2 < y_3$. Want to find $(p_{y_1,k}, p_{y_2,k}, p_{y_3,k})$ such that

$$\begin{pmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ y_1^2 & y_2^2 & y_3^2 \end{pmatrix} \begin{pmatrix} p_{y_1,k} \\ p_{y_2,k} \\ p_{y_3,k} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\rho}{\lambda} \\ k \end{pmatrix}$$

Solving this, we get:

$$\begin{aligned} p_{y_1,k} &= \frac{\lambda y_3 y_2 - \rho y_3 - \rho y_2 + \lambda k}{\lambda(y_3 - y_1)(y_2 - y_1)} \\ p_{y_2,k} &= \frac{-\lambda y_1 y_3 + \rho y_1 + \rho y_3 - \lambda k}{\lambda(y_2 - y_1)(y_3 - y_2)} \\ p_{y_3,k} &= \frac{y_1 y_2 \lambda - \rho y_1 - \rho y_2 + k\lambda}{\lambda(y_3 - y_1)(y_3 - y_2)} \end{aligned}$$

Must have: $(1)(\rho/\lambda)^2 < k \quad (2) y_1 < \rho/\lambda < y_3$

Fix y_1. $p_{y_2,k}, p_{y_3,k} > 0 \Rightarrow y_2 < (k\lambda - \rho y_1)/(\rho - \lambda y_1) < y_3$

Fix $y_2 < \rho/\lambda$. $p_{y_1,k} > 0 \Rightarrow y_3 < (k\lambda - \rho y_2)/(\rho - \lambda y_2)$

Fix $y_2 > \rho/\lambda$. $p_{y_1,k} > 0 \Rightarrow y_3 > (\rho y_2 - \lambda k)/(y_2\lambda - \rho)$

Another issue here is that for computational purposes the search for maximum needs to be restricted to finite limits. For stock prices the lower limit is always zero, since the stock price cannot be negative. Theoretically there is no upper limit. So we derive in Appendix E.1 the probability of the stock price lying below some given bounds. Then, given a probability p close to 1, we obtain the corresponding upper bound on the stock price and carry out the computation by restricting the stock price to be below that bound. This gives bounds on the option prices that hold with probability p.

4.2.2 *Distribution of* ξ_T

Let

$$\begin{aligned} P_{n,k}(t) &= \mathrm{P}(N_t = n | N_0 = k) \\ Q_{m,k}(t) &= \mathrm{P}(\xi_t = m | N_0 = k) \\ P_{n,m,k}(t) &= \mathrm{P}(N_t = n | \xi_t = m, N_0 = k) \end{aligned}$$

Claim 4.2.2.1. $Q_{m,k}(t) = \frac{(\frac{k\lambda}{\rho}+m-1)!}{(\frac{k\lambda}{\rho}-1)!} e^{-k\lambda t} \frac{(1-e^{-\rho t})^m}{m!}$

Proof. For $m \geq 1$,

$$\begin{aligned} Q_{m,k}(t+dt) &= Q_{m-1,k}(t) \sum_n P_{n,m-1,k}(t) n\lambda dt + Q_{m,k}(t)(1 - \sum_n P_{n,m,k}(t) n\lambda dt) \\ &= Q_{m-1,k}(t) \mathrm{E}[N_t | \xi_t = m-1, N_0 = k] \lambda dt \\ &\quad + Q_{m,k}(t)(1 - \mathrm{E}[N_t | \xi_t = m, N_0 = k] \lambda dt) \\ &= Q_{m-1,k}(t)(k + \frac{(m-1)\rho}{\lambda}) \lambda dt + Q_{m,k}(t)\{1 - (k + \frac{m\rho}{\lambda}) \lambda dt\} \end{aligned}$$

$$Q'_{m,k}(t) = Q_{m-1,k}(t)(k + \frac{(m-1)\rho}{\lambda})\lambda - Q_{m,k}(t)(k + \frac{m\rho}{\lambda})\lambda \quad Q_{m,k}(0) = 0$$

$$Q_{0,k}(t+dt) = Q_{0,k}(t)(1 - \sum_n \mathrm{P}_{n,0}(t) n\lambda dt) = Q_{0,k}(t)\{1 - k\lambda dt\}$$

$$Q'_{0,k}(t) = -Q_{0,k}(t) k\lambda \quad Q_{0,k}(0) = 1$$

It can be easily verified that the proposed expression for $Q_{m,k}(t)$ satisfies the conditions derived above. □

For computational purposes, we have to restrict to finite values of ξ_T. It is shown in Appendix E.1 that for every c, the number of jumps in finite time is finite almost surely.

4.3 Bounds on option prices for a Discrete Time Approximation

The algorithm described in section 4.2 requires intensive computation. In this section we propose a discrete time jump model which has the same convergence properties as the continuous time jump model under consideration. It is possible to derive approximate bounds on option prices very easily for the discrete time model. This saves time and cost for computation. In section 9.1 we compute the exact bounds on prices using the continuous-time model and approximate bounds using the discrete-time model and see the amount of loss of precision due to the approximation.

Suppose the stock price process S_t moves on a grid of size $1/n$, making a jump of size Y_i/n where Y_i is an integer valued random variable, at each time point $i \in \{1/m, 2/m, \ldots, Tm/m\}$. Let $\mathcal{F}_i = \sigma\{S_{j/m} : 0 \leq j \leq i\}$

$$S_t = S_0 + \sum_{i=1}^{[tm]} \frac{Y_i}{n}$$

A similar proof as that in section 3.2 can be done to show that the moment conditions for convergence of this model to geometric Brownian motion are:

$$\mathrm{E}(Y_i|\mathcal{F}_{i-1}) = \frac{\rho n}{m} S_{(i-1)/m} \tag{F1}$$

$$\mathrm{E}(Y_i^2|\mathcal{F}_{i-1}) = \frac{\lambda n^2}{m} S_{(i-1)/m}^2 \tag{F2}$$

We want to maximize $\mathrm{E}(S_t - K)_+ = \mathrm{E}(S_0 - K + \sum_{i=1}^{[Tm]} Y_i/n)_+$ under (F1) - (F2). This involves sequential maximization in $T \times m$ stages. We show in Proposition 4.3.0.1 that these conditions imply the more general conditions:

$$\mathrm{E}(\sum_{i=1}^{[Tm]} Y_i) = nS_0\{(1 + \frac{\rho}{m})^{Tm} - 1\} = x_m \tag{G1}$$

$$\mathrm{E}[(\sum_{i=1}^{[Tm]} Y_i)^2] = n^2 S_0^2 [1 + (1 + \frac{\lambda}{m} + 2\frac{\rho}{m})^{Tm} - 2(1 + \frac{\rho}{m})^{Tm}] = y_m \tag{G2}$$

The maximum obtained under these more general conditions is larger than the exact maximum obtained under the conditions (F1) - (F2). However in this case the maximization is a one-stage procedure and is computationally much less intense. The maximum is attained by a 3-point distribution which can be computed easily.

Proposition 4.3.0.1. *(F1) - (F2)* $\Longrightarrow$ *(G1) - (G2)*

Proof. Let $f_k = \mathrm{E}[\sum_{i=1}^{k} Y_i]$ and $g_k = \mathrm{E}[(\sum_{i=1}^{k} Y_i)^2]$

$$\begin{aligned} f_k &= \mathrm{E}[\sum_{i=1}^{k-1} Y_i + \frac{\rho n}{m} S_{(k-1)/m}] = f_{k-1} + \frac{\rho n}{m} S_0 + \frac{\rho}{m}\mathrm{E}[\sum_{i=1}^{k-1} Y_i] \\ &= (1+\frac{\rho}{m}) f_{k-1} + \frac{\rho n}{m} S_0 = a + b f_{k-1} \end{aligned}$$

where $a = \rho n S_0/m$ and $b = 1 + \rho/m$. It is shown in Appendix D that this implies

$$f_k = a\frac{b^k - 1}{b-1} = nS_0\left((1+\frac{\rho}{m})^k - 1\right)$$

$$\begin{aligned} g_k &= \mathrm{E}[(\sum_{i=1}^{k-1} Y_i)^2 + 2(\sum_{i=1}^{k-1} Y_i) \times \mathrm{E}(Y_k|\mathcal{F}_{k-1}) + \mathrm{E}(Y_k^2|\mathcal{F}_{k-1})] \\ &= g_{k-1} + \mathrm{E}[2(\sum_{i=1}^{k-1} Y_i) \times \frac{\rho n}{m} S_{(k-1)/m} + \frac{\lambda n^2}{m} S^2_{(k-1)/m}] \\ &= g_{k-1} + 2\frac{\rho n}{m} S_0 \mathrm{E}[(\sum_{i=1}^{k-1} Y_i)] + 2\frac{\rho}{m}\mathrm{E}[(\sum_{i=1}^{k-1} Y_i)^2] + \frac{\lambda n^2}{m}\mathrm{E}[(S_0 + \sum_{i=1}^{[k-1]} \frac{Y_k}{n})^2] \\ &= g_{k-1} + 2\frac{\rho n}{m} S_0 f_{k-1} + 2\frac{\rho}{m} g_{k-1} + \frac{\lambda}{m}(n^2 S_0^2 + 2nS_0 f_{k-1} + g_{k-1}) \\ &= (1 + 2\frac{\rho}{m} + \frac{\lambda}{m}) g_{k-1} + \frac{\lambda n^2}{m} S_0^2 + 2(\frac{\rho}{m} + \frac{\lambda}{m}) n^2 S_0^2 \{(1+\frac{\rho}{m})^{k-1} - 1\} \\ &= a + b g_{k-1} + c d^{k-1} \end{aligned}$$

where $a = n^2 S_0^2[\frac{\lambda}{m} - 2(\frac{\lambda}{m} + \frac{\rho}{m})]$, $b = 1 + \frac{\lambda}{m} + 2\frac{\rho}{m}$, $c = 2nS_0^2(\frac{\lambda}{m} + \frac{\rho}{m})$, $d = 1 + \frac{\rho}{m}$.

It is shown in Appendix D that this implies

$$
\begin{aligned}
g_k &= a\frac{b^k-1}{b-1} + c\frac{b^k-d^k}{b-d} \\
&= -nS_0^2(\frac{\lambda}{m}+2\frac{\rho}{m})\frac{b^k-1}{\frac{\lambda}{m}+2\frac{\rho}{m}} + 2n^2S_0^2(\frac{\lambda}{m}+\frac{\rho}{m})\frac{b^k-d^k}{\frac{\lambda}{m}+\frac{\rho}{m}} \\
&= n^2S_0^2[1+(1+\frac{\lambda}{m}+2\frac{\rho}{m})^k - 2(1+\frac{\rho}{m})^k]
\end{aligned}
$$

□

Chapter 5

BIRTH AND DEATH MODEL

5.1 The model

As noted in chapter 1, the general jump models described so far, in spite of being good descriptions of the stock price and useful for pricing options, have one serious drawback. Self financing strategies for derivative securities cannot be created under these models. Birth-death processes, on the other hand, are pure jump models and have the virtue that they allow for derivative securities hedging. In the next few chapters we shall develop the theory of pricing and hedging for a class of birth and death processes that converge in the limit to geometric Brownian motion.

Let us suppose that the stock price S_t is a pure jump process with jumps of size $\pm c$. This implies that the process moves on a grid of resolution c and $N_t = S_t/c$ is a birth and death process. The jumps of the N_t process have random size Y_t which is a binary variable taking values ± 1 and the probability that $Y_t = 1$ is denoted by p_{t,N_t}. We suppose that there is a risk-free interest rate ρ_t and the intensity of jumps is $N_t^2 \lambda_t$ where the *rate* λ_t is a non-negative stochastic process. A more natural assumption would be to take the intensity to be proportional to N_t. However, as we showed in section 3.1, such linear intensity processes converge to a deterministic process as $c \to 0$. We show in what follows that the quadratic intensity model (intensity of jumps proportional to N_t^2) converges to geometric Brownian motion as $c \to 0$. In order to keep the process away from zero, we introduce the condition:

$$\text{When } N_t = 1, Y_t \text{ takes values 0 and 1 with probabilities } p_{t,1} \text{ and } 1 - p_{t,1} \tag{5.1}$$

We now have the following result.

Proposition 5.1.0.2. *Let* $N_t^{(n)}$ *be an integer valued jump process, the jump time* $\xi_t^{(n)}$ *following a counting process with rate* $N_t^{(n)2}\sigma_t^2$ *and the random jump size* $Y_t^{(n)}$ *which is a binary variable taking values* ± 1 *and probability that* $Y_t = 1$ *is* p_{t,N_t} *and satisfying condition (5.1) and assumptions* (H1)-(H2). Then $X_t^{(n)} = \ln(N_t^{(n)}/n)$ converges in distribution to X_t, a continuous Gaussian Martingale with characteristics $(\int_0^t(\rho_u - \frac{1}{2}\sigma_u^2)du, \int_0^t \sigma_u^2 du, 0)$ if $p_{t,N_t} = \frac{1}{2}(\frac{\rho_t}{N_t\sigma_t^2} + 1)$ and $p_{t,1} = \frac{\rho_t - \frac{\sigma_t^2}{2}}{\sigma_t^2 \log(2)}$

Proof. The proof is given in Appendix B. ☐

Unlike the general jump model, in this case the jump distribution is completely specified by the martingale condition and it satisfies all the regularity conditions. So we do not need to specify them separately. The assumptions are:

$$\sum_{\tau_i \le t} \left| \log\left(1 + \frac{Y_{\tau_i} t^{(n)}}{N_{\tau_i}^{(n)2}}\right) \right|^4 \xrightarrow{\text{P}} 0 \tag{H1}$$

$$\int_0^t N_u^2 \sigma_u^2 du \quad \text{is finite a.s.} \tag{H2}$$

Suppose for each n, the stock price $S_t^{(n)} = N_t^{(n)}/n$ where the process $N_t^{(n)}$ is described in Proposition 5.1.0.2. So the grid size c is $1/n$. We assume initial stock price $S_0^{(n)}$ is the same for all n. As $n \to \infty$, by Proposition 5.1.0.2, the sequence of random processes $X_t^{(n)} = \ln(S_t^{(n)})$ converge in distribution to X_t, a continuous Gaussian Martingale with characteristics $(\int_0^t(\rho_u - \frac{1}{2}\sigma_u^2)du, \int_0^t \sigma_u^2 du, 0)$. Since exp is a continuous function, $S^{(n)} = \exp(X^{(n)})$ converge in law to $\exp(X)$. The stochastic differential equation of X is:

$$d(X_t) = (\rho_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t \quad \text{where } W_t \text{ is standard Weiner process} \tag{5.2}$$

By Ito's formula,

$$\begin{aligned} d(S_t) = d(\exp(X_t)) &= S_t[(\rho_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t] + \frac{1}{2}S_t\sigma_t^2 dt \\ &= S_t\rho_t dt + S_t\sigma_t dW_t \end{aligned} \tag{5.3}$$

In chapter 6 we consider processes with intensity $\lambda_t N_t^2$ where λ_t is a constant. In chapter 7, we consider the case of λ_t being a stochastic process and N_t is a birth and death process conditional on the λ_t process. This can be formally carried out by letting N_t be the integral of the Y_t process with respect to the random measure that has intensity $\lambda_t N_t^2$(see, *e.g*, Ch. II.1.d of Jacod and Shiryaev(1987)).

5.2 Edgeworth Expansion for Option Prices

Let us define

$$\begin{aligned} X_t^{(n)} &= ln(\frac{N_t^{(n)}}{n}) \\ X_t^{*(n)} &= X_t^{(n)} - \int_0^t [p_{u,N_u}\log(1+\frac{1}{N_u}) + (1-p_{u,N_u})\log(1-\frac{1}{N_u})]N_u^2\sigma_u^2 du - X_0^{(n)} \\ \text{where} \quad & p_{t,N_t} = \frac{1}{2}(1+\frac{\rho_t}{N_t\sigma_t^2}) \end{aligned}$$

Let $\mathcal{C}$ be the class of functions g that satisfy the following: (i) $\int \mid \hat{g}(x) \mid dx < \infty$, uniformly in C, and $\{\sum_u x_u^2 \hat{g}(x), g \in C\}$ is uniformly integrable (here, $\hat{g}$ is the

Fourier transform of g, which must exist for each $g \in C$); or (ii) $g(x) = f(z^i x_i)$, with $\sum_i z^i z^i$, f nd f'' bounded, uniformy in $\mathcal{C}$, and with $\{f'' : g \in \mathcal{C}\}$ equicontinuous almost everywhere (under Lebesgue measure). Under assumptions (I1) and (I2) stated in Appendix C, for any $g \in \mathcal{C}$,

$$\mathrm{E}g(X_T^{*(n)}) \quad = \quad \mathrm{E}g(N(0, \lambda T)) + o(1/n)$$

The details are given in Appendix C.

Chapter 6

PRICING AND HEDGING OF OPTIONS WHEN THE INTENSITY RATE IS CONSTANT

Let us first consider the case when λ is constant. For any λ, let $\mathcal{P}_\lambda$ be the measure associated to a birth-death process with event rate λN_t^2 and probability of birth $p_{t,N_t} = \frac{1}{2}(1 + \rho_t/(\lambda N_t))$.

6.1 Risk-neutral distribution

Starting from no arbitrage assumptions, the fundamental theorem of asset pricing asserts the existence of an equivalent measure $\mathcal{P}^*$, called the risk neutral measure, such that discounted prices of the stock and all traded derivative securities are (local) martingales under this measure. The history of this theorem goes back to Harrison and Kreps(1979). Since then many authors have made contributions to improve the understanding of this theorem under various conditions eg Duffie and Huang(1986), Delbaen and Schachermayer(1994, 1998).

In this section we discuss the conditions of this theorem for the specific model under consideration. In proposition 6.1.0.3 we show that for any λ^*, the measure $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda^*}$ are equivalent and the discounted stock price is a $\mathcal{P}_{\lambda^*}$ martingale. Thus there are infinitely many equivalent martingale measures and the model is incomplete. We have to introduce another security to complete the model. If we assume that there is a market traded derivative security with price process P_t, then proposition 6.1.0.4

gives conditions for existence of $\mathcal{P}^*$ equivalent to $\mathcal{P}_\lambda$ such that under $\mathcal{P}^*$ discounted S_t and P_t are martingales. Now if we assume that this $\mathcal{P}^*$ is a birth and death process with constant intensity rate λ^*, then proposition 6.1.0.5 shows that this λ^* is unique and gives the mathematical expression for λ^*. Proposition 6.1.0.6 and the following argument shows that if discounted S_t and P_t are $\mathcal{P}_{\lambda^*}$ martingales, then the price of any integrable contingent claim is $\mathcal{P}_{\lambda^*}$ martingale.

Note that if we take σ_t^2 in proposition 5.1.0.2 to be equal to λ^*, then even under the risk neutral measure, the stock price process converges in distribution to geometric Brownian motion as the grid-size goes to zero.

Proposition 6.1.0.3. *For any $\tilde{\lambda}$, the probability measures $\mathcal{P}_{\tilde{\lambda}}$ and $\mathcal{P}_\lambda$ are mutually absolutely continuous and the discounted security price is a $\mathcal{P}_{\tilde{\lambda}}$ martingale.*

Proof. Note that all birth-death processes are supported on the class of step functions that are right continuous and have left limits(*r.c.l.l.*) with jumps of size ± 1 on the non-negative integers. Uniqueness of the associated measures corresponding to a jump intensity rate and probability of birth is a consequence of e. g. Thm 18.4/5 in Lipster and Shiryaev(1978). Hence the measures associated with all birth death processes are mutually absolutely continuous. As a consequence of Thm 19.7 of Lipster and Shiryaev(1978) we can even give the explicit form of $\mathcal{P}_{\tilde{\lambda}}$ via its Radon-Nikodym derivative with respect to $\mathcal{P}$ as:

$$\frac{d\mathcal{P}_{\tilde{\lambda}}}{d\mathcal{P}} = \exp\left(\int_0^T \ln(\frac{\tilde{\lambda}_t \tilde{p}_{t,N_t}}{\lambda_t p_{t,N_t}}) dN_{1t} + \int_0^T \ln(\frac{\tilde{\lambda}_t(1-\tilde{p}_{t,N_t})}{\lambda_t(1-p_{t,N_t})}) dN_{2t} - \int_0^T (\tilde{\lambda}_t - \lambda_t) N_t^2 dt\right)$$

where $dN_{1t} = I_{Y_t=+1} dN_t$ and $dN_{2t} = -I_{Y_t=-1} dN_t$ with $N_{i0} = 0$. N_{1t}, N_{2t} are point processes with intensity $\lambda_t p_{t,N_t} N_t^2$ and $\lambda_t(1-p_{t,N_t})N_t^2$ respectively and $dN_t = dN_{1t} - dN_{2t}$. Let $\tilde{\lambda}_{1t} = \tilde{\lambda}\tilde{p}_{t,N_t}N_t^2$ and $\tilde{\lambda}_{2t} = \tilde{\lambda}(1-\tilde{p}_{t,N_t})N_t^2$. Then

$Q_{it} = N_{it} - \int_0^t \tilde{\lambda}_{is} N_s^2 ds$ is the compensated point process associated with N_i under $\mathcal{P}_{\tilde{\lambda}}$.

$$\begin{aligned} dN_t &= dN_{1t} - dN_{2t} \\ &= dQ_{1t} - dQ_{2t} + (\tilde{\lambda}_{1t} - \tilde{\lambda}_{2t}) N_t^2 dt \\ &= dQ_{1t} - dQ_{2t} + (2\tilde{p}_{t,N_t} - 1)\tilde{\lambda}_t N_t^2 dt \\ &= dQ_{1t} - dQ_{2t} + \rho_t N_t dt \end{aligned}$$

So $\exp\{-\int_0^t \rho_s ds\} N(t)$ is a martingale with respect to $\mathcal{P}_{\tilde{\lambda}}$. □

Proposition 6.1.0.4. *Let us assume that there is a market traded derivative security with price process P_t. Assume that there is no global free lunch, then there exists a measure $\mathcal{P}^*$ which is equivalent to $\mathcal{P}$ and under which discounted S_t and P_t are martingales.*

Proof. Refer to Kreps(1981) □

Proposition 6.1.0.5. *Assume that $\mathcal{P}^*$ in Proposition 6.1.0.4 is a birth death process with quadratic intensity with constant rate λ^*. Then λ^* is unique and equals*

$$\frac{1}{T}\left[\ln\left(\mathrm{E}_{\mathcal{P}^*}\left(\exp\{-\int_0^T \rho_s ds\} S_T^2\right)\right) - \ln S_0^2 - \int_0^T \rho_s ds\right]$$

Proof. $dN_t^2 = 2N_t dN_t + \frac{1}{2} 2d < N, N >_t$

$dN_t = dQ_{1t} - dQ_{2t} + \rho_t N_t dt$

$d < N, N >_t = 1.\lambda^* N_t^2 dt$

So, $dN_t^2 = 2N_t dQ_{1t} - 2N_t dQ_{2t} + (2\rho_t + \lambda^*) N_t^2 dt$

Hence $\exp\{-\int_0^T 2\rho_s ds - \lambda^* T\} S_T^2$ is a $\mathcal{P}_{\lambda^*}$ martingale.

$$\begin{aligned}
&\mathrm{E}_{\mathcal{P}_{\lambda^*}}\left(\exp\{-\int_0^T \rho_s ds\} S_T^2\right) \\
&= \mathrm{E}_{\mathcal{P}_{\lambda^*}}\left(\exp\{-\int_0^T 2\rho_s ds - \lambda^* T\} S_T^2\right) \exp\{\int_0^T \rho_s ds + \lambda^* T\} \\
&= S_0^2 \exp\{\int_0^T \rho_s ds + \lambda^* T\} = \mathrm{E}_{\mathcal{P}^*}\left(\exp\{-\int_0^T \rho_s ds\} S_T^2\right)
\end{aligned}$$

and this is unique λ^* with this property.

$$\begin{aligned}
&\mathrm{E}_{\mathcal{P}^*}\left(\exp\{-\int_0^T \rho_s ds\} S_T^2\right) \\
&> \exp\{\int_0^T \rho_s ds\} \left[\mathrm{E}_{\mathcal{P}^*}\left(\exp\{-\int_0^T \rho_s ds\} S_T\right)\right]^2 \\
&= \exp\{\int_0^T \rho_s ds\} S_0^2
\end{aligned}$$

$$\text{Hence} \quad \lambda^* = \frac{1}{T}\left[\ln\left(\mathrm{E}_{\mathcal{P}^*}\left(\exp\{-\int_0^T \rho_s ds\} S_T^2\right)\right) - \ln S_0^2 - \int_0^T \rho_s ds\right] > 0$$

□

Proposition 6.1.0.6. *Assume that there is a λ^* such that discounted S_t and P_t are $\mathcal{P}_{\lambda^*}$ martingales. Then the price associated with any integrable contingent claim X at time t is $\pi(t) = \mathrm{E}_{\mathcal{P}_{\lambda^*}}\left(\exp\{-\int_t^T \rho_s ds\} X | \mathcal{F}_t\right)$*

Proof. It follows from Jacod(1975) that any martingale Y_t can be represented as :

$$Y_t = Y_0 + \int_o^t f(s,1) dQ_{1s} + \int_o^t f(s,2) dQ_{2s} \tag{6.1}$$

where f is a predictable process.

Let $Z_{1t} = \exp\{-\int_0^t \rho_s ds\} N_t,\ Z_{2t} = \exp\{-\int_0^t \rho_s ds\} P_t$

$dZ_{1t} = \exp\{-\int_0^t \rho_s ds\}(dN_{1t} - dN_{2t} - \rho_t N_t dt)$

$dZ_{2t} = g(t,1)dQ_{1t} + g(2,t)dQ_{2t}$ from (6.1)

From here, using the relationship between Q_{it} and N_{it}, we get:

$$\begin{aligned} dQ_{1t} &= \left(\frac{\exp\{\int_0^t \rho_s ds\} g(t,2) dZ_{1t} - dZ_{2t}}{g(t,2) - g(t,1)}\right) \\ dQ_{2t} &= \left(\frac{-\exp\{\int_0^t \rho_s ds\} g(t,1) dZ_{1t} + dZ_{2t}}{g(t,2) - g(t,1)}\right) \end{aligned}$$

So Y_t can be expressed as:

$$Y_t = Y_0 + \int_o^t \hat{f}(s,1) dZ_{1s} + \int_o^t \hat{f}(s,2) dZ_{2s}$$

Hence by Thm 3.35 of Harrison and Pliska(1980), the model is now complete and the price associated with any integrable contingent claim X at time t is $\pi(t) = \mathrm{E}_{\mathcal{P}_{\lambda^*}}(e^{-\rho(T-t)} X | \mathcal{F}_t)$ □

6.2 Hedging

We have shown that the market is complete when we add a market traded derivative security. So we can hedge an option by trading the stock, the bond and another option. In this section we obtain the hedge ratios by forming a self-financing risk-less portfolio with the stock, the bond and two options.

Let $F_2(x,t), F_3(x,t)$ be the prices of two options at time t when the price of the stock is cx. Let $F_0(x,t) = B_0 \exp\{-\int_0^t \rho_s ds\}$ be the price of the bond and $F_1(x,t) = cx$ be the price of the stock. Assume F_i are continuous in both arguments.

We shall construct a self financing risk-less portfolio

$$V(t) = \sum_{i=0}^{3} \phi^{(i)}(t) F_i(x,t)$$

Let $u^{(i)}(t) = \frac{\phi^{(i)}(t)F_i(x,t)}{V(t)}$ be the proportion of wealth invested in asset i.

$$\sum u^{(i)} = 1 \tag{6.2}$$

Since V_t is self financing,

$$\begin{aligned}\frac{dV(t)}{V(t)} &= \sum_{i=0}^{3} u^{(i)}(t) \frac{dF(x,t)}{F(x,t)} \\ &= u^{(0)}(t)\rho_t dt + u^{(1)}(t)\frac{1}{cx}(dN_{1t} - dN_{2t}) \\ &\quad + \sum_{i=2}^{3} u^{(i)}(t)(\alpha_{F_i}(x,t)dt + \beta_{F_i}(x,t)dN_{1t} + \gamma_{F_i}(x,t)dN_{2t})\end{aligned}$$

V_t is risk-less implies

$$u^{(1)}(t)\frac{1}{cx} + \sum_{i=2}^{3} u^{(i)}(t)\beta_{F_i}(x,t) = 0 \tag{6.3}$$

$$-u^{(1)}(t)\frac{1}{x} + \sum_{i=2}^{3} u^{(i)}(t)\gamma_{F_i}(x,t) = 0 \tag{6.4}$$

The no arbitrage assumption implies

$$u^{(0)}(t)\rho_t + \sum_{i=2}^{3} u^{(i)}(t)\alpha_{F_i}(x,t) = \rho_t \tag{6.5}$$

Solving equations (6.2)-(6.5), we get the hedge ratios as:

$$
\begin{aligned}
u^{(2)} &= [(1 - \frac{\alpha_{F_2}}{\rho} - x\beta_{F_2}) - \frac{\gamma_{F_2} + \beta_{F_2}}{\gamma_{F_3} + \beta_{F_3}}(1 - \frac{\alpha_{F_2}}{\rho} - x\beta_{F_2})]^{-1} \\
u^{(3)} &= [(1 - \frac{\alpha_{F_3}}{\rho} - x\beta_{F_3}) - \frac{\gamma_{F_3} + \beta_{F_3}}{\gamma_{F_2} + \beta_{F_2}}(1 - \frac{\alpha_{F_3}}{\rho} - x\beta_{F_3})]^{-1} \\
u^{(0)} &= -\frac{1}{\rho_t}(u^{(2)}\alpha_{F_2} + u^{(3)}\alpha_{F_3}) \\
u^{(1)} &= -x(u^{(2)}\beta_{F_2} + u^{(3)}\beta_{F_3})
\end{aligned}
$$

Chapter 7

STOCHASTIC INTENSITY RATE

Now we consider the case where the unobserved intensity rate λ_t is a stochastic process. We first assume a two state Markov model for λ_t. In section 8.5 we describe how we can have similar results for other models on λ_t.

Suppose there is an unobserved state process θ_t which takes 2 values, say 0 and 1. The transition matrix is $\mathbf{Q}$. When $\theta_t = i$, $\lambda_t = \lambda_i$. Jump process associated with θ_t is ζ_t.

Let us denote by $\{\mathcal{G}_t\}$ the complete filtration $\sigma(S_u, \lambda_u, 0 \leq u \leq t)$ and by $\mathcal{P}$ the class of probability measures on $\{\mathcal{G}_t\}$ associated with the process (S_t, λ_t).

7.1 Risk-neutral distribution

Let us assume that the risk-neutral measure is the measure associated with a birth-death process with jump intensity $\lambda_t^* N_t^2$ and probability of birth $p_t^* = \frac{1}{2}(1 + \frac{\rho_t}{\lambda_t^* N_t})$ where λ_t^* is a Markov process with state space $\{\lambda_1, \lambda_2\}$.

As noted in Section 6.1, the model is incomplete and we need to introduce a market traded option to complete the model. In order to determine the risk-neutral parameters, we have to equate the observed price of a market traded option to its expected price under the model. We get two different values of the expected price under the two values of $\theta(0)$. The θ process is unobserved. So there is no way of determining which value to use. We cannot invert an option to get $\theta(0)$ either, because it takes two discrete values and does not vary over a continuum.

We need to introduce $\pi_i(t) = \mathrm{P}(\theta_t = i|\mathcal{F}_t)$ where $\mathcal{F}_t = \sigma(S_u, 0 \leq u \leq t)$

Let $F_i(x,t) := \mathrm{E}_{\hat{\Theta}}(X|\mathcal{G}_t)$ when the stock price is cx and $\lambda_t^* = \lambda_i$.

Let $G(x,t) := \mathrm{E}_{\hat{\Theta}}(X|\mathcal{F}_t) = \pi_0(t)F_0(x,t) + \pi_1(t)F_1(x,t)$

As shown in Snyder(1973), under any $\hat{P} \in \mathcal{P}$ the π_{it} process evolves as:

$$d\pi_{1t} = a(t)dt + b(t,1)dN_{1t} + b(t,2)dN_{2t}$$

where $a(t)$ and $b(t,i)$ are $\mathcal{F}_t$ adapted processes.

Proposition 7.1.0.7. *dG/G=$\tilde{\alpha_F}dt + \tilde{\beta_F}dN_{1t} + \tilde{\gamma_F}dN_2$ where*

$$\tilde{\alpha_F}(x,t) = \frac{\pi_{0t}\frac{\partial F_0}{\partial t} + \pi_{1t}\frac{\partial F_1}{\partial t} + a(t)(F_0 - F_1)}{\pi_{0t}F_0 + \pi_{1t}F_1}$$

$$\tilde{\beta_F} = \frac{\pi_{0t}\beta_{F_0}F_0 + \pi_{1t}\beta_{F_1}F_1 + b(t,1)(F_0(1+\beta_{F_0}) - F_1(1+\beta_{F_1}))}{\pi_{0t}F_0 + \pi_{1t}F_1}$$

$$\tilde{\gamma_F} = \frac{\pi_{0t}\gamma_{F_0}F_0 + \pi_{1t}\gamma_{F_1}F_1 + b(t,-1)(F_0(1+\gamma_{F_0}) - F_1(1+\gamma_{F_1}))}{\pi_{0t}F_0 + \pi_{1t}F_1}$$

Proof.

$$\begin{aligned}
dG &= \pi_0 dF_0 + \pi_1 dF_1 + F_0 d\pi_0 + F_1 d\pi_1 + \text{covariance term} \\
&= \pi_0 dF_0 + \pi_1 dF_1 + (F_0 - F_1)d\pi_0 + d[F_0 - F_1, \pi_0] \\
&= -(\pi_0 \frac{\partial F_0}{\partial t} + \pi_1 \frac{\partial F_1}{\partial t})dt \\
&\quad +(\pi_0\beta_{F_0}F_0 + \pi_1\beta_{F_1}F_1)dN_{1t} + (\pi_0\gamma_{F_0}F_0 + \pi_1\gamma_{F_1}F_1)dN_{2t} \\
&\quad\quad +(F_0 - F_1)(a(t)dt + b(t,1)dN_{1t} + b(t,-1)dN_{2t}) \\
&\quad\quad +(\beta_{F_0}F_0 - \beta_{F_1}F_1)b(t,1)dN_{1t} + (\gamma_{F_0}F_0 - \gamma_{F_1}F_1)b(t,-1)dN_{2t} \\
&= -\tilde{\alpha}dt + \tilde{\beta}dN_{1t} + \tilde{\gamma}dN_{2t}
\end{aligned}$$

□

Proposition 6.1.0.6 holds now under $P_{\hat{\Theta}}$. We still need one market traded option and the stock to hedge an option. But to get the hedge ratios, we need $\pi^*(t), a(t), b(t)$.

7.2 Inverting options to infer parameters

Inference on the parameter $\pi^*(t)$ can be done in the risk-neutral setting by inverting options at each point of time. Besides involving huge amount of computations this also has the following theoretical drawbacks:

- The hedge ratios involve $a(t), b(t)$. This implies that we need them to be predictable. But if we have to invert an option to get them, then we need to observe the price at time t to infer π_t and from there to get a_t and b_t. So they are no more predictable.

- Options are not as frequently traded as stocks and hence option prices are not as reliable as stock prices. Thus, inferring $\pi(t)$ at each time point t by inverting options will give incorrect prices and lead to arbitrage.

So we base our updates only on the stock prices. Inference on the initial parameter $\pi(0)$, the transition rates q_{01}, q_{10} and the jump rates λ_0, λ_1 is done under the risk-neutral measure based on stock and option prices. However subsequent inference on the process π_t is done based only on the stock price process.

Chapter 8

BAYESIAN FRAMEWORK

8.1 The Posterior

As shown in Yashin(1970) and Elliott et al(1995), the posterior of $\pi_j(t)$ is given by:

$$\begin{aligned}\pi_j(t) =& \pi_j(0) + \int_0^t \sum_i q_{ij}\pi_i(u)du + \int_0^t \pi_j(u)(\bar{\lambda}(u) - \lambda_j)N_u^2 du \\ &+ \sum_{0<u<t} \pi_j(u-)\left(\frac{\lambda_j p_{\lambda_j}(S_u - \to S_u)}{\sum_i \pi_i(u)\lambda_i p_{\lambda_i}(S_u - \to S_u)} - 1\right)\end{aligned}$$

where $\bar{\lambda}(t) = \sum_i \pi_i(t)\lambda_i$

Thus, $a_j(u) = \sum_i q_{ij}\pi_i(u) + \int_0^t \pi_j(u)(\bar{\lambda}(u) - \lambda_j)N_u^2$,

and $b_j(u) = \pi_j(u-)\left(\frac{\lambda_j p_{\lambda_j}(S_{u-} \to S_u)}{\sum_i \pi_i(u)\lambda_i p_{\lambda_i}(S_{u-} \to S_u)} - 1\right)$

8.2 Hedging

Same results as in the fixed λ case holds with α, β, γ replaced by $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$. In this setting we need one option and the stock to hedge an option and do not need to invert at all time points as would be case if we did not use the posterior.

8.3 Pricing

In section 10.4.2 we describe the method of obtaining the risk-neutral parameters $\hat{\Theta}$ by equating the observed price of an option to $G(N_0, 0)$. Here is the algorithm to get $\mathrm{E}(\exp\{-\int_0^T \rho_s ds\}X|\mathcal{F}_0)$ under fixed values of $\mathbf{Q}, \lambda_0, \lambda_1, \pi_0(0)$?

- Fix $\theta_0 = i$
- Generate the θ process.
- Generate ξ_t, the process of waiting times as non-homogeneous Poisson process with intensity rate λ_{θ_t}.
- At each event time S_t jumps by $\pm c$ with probability p_{t,N_t} and $1 - p_{t,N_t}$.
- Get the expectation under $\theta_0 = i$ for $i = 0, 1$.
- Now take the average of these with respect to π_0.

The only difficult step here is generating the ξ_t process. This is done as follows:

- Generate T_0 from $\mathrm{Exp}(\lambda_{\theta_0} N_0^2)$
- Let $\tau_0 = \inf\{t : \theta_t \neq \theta_0\}$
- If $T_0 < \tau_0$, jump at time T_0.
- Otherwise, generate T_1 from $\mathrm{Exp}(\lambda_{\theta_{\tau_0}} N_{\tau_0}^2)$
- $\tau_1 = \inf\{t : \theta_t \neq \theta_{\tau_0}\}$
- If $T_1 < \tau_1$, jump at time $\tau_0 + T_1$
- Continue.

This is justified by memorylessness.

$$\text{Proof: } \mathrm{P}(\text{jump at } a, \tau_0 < a < \tau_1) = \mathrm{P}(T_0 > \tau_0 \text{ and } T_1 = a < \tau_1)$$
$$= e^{-\tau_0 \lambda_{\theta_0} N_0^2} \lambda_{\theta_{\tau_0}} N_{\tau_0}^2 e^{-a\lambda_{\theta_{\tau_0}} N_{\tau_0}^2}$$
$$= \lambda_{\theta_a} N_a^2 e^{-\int_0^a \lambda_{\theta_t} N_t^2 dt}$$

8.4 Reducing number of options required

We need to infer 5 parameters $q_{01}, q_{10}, \lambda_0, \lambda_1, \pi_0(0)$. This involves inverting the prices of 5 options with different maturities at time 0. Different strike prices for the same maturity do not give additional information for the model. Many options do not have so many maturities. So it would be useful if we could reduce the number of maturities and use a fewer number of options at various time points. Using the filtering equations it is in fact possible to infer the required quantities by using one option at 5 points of time close to 0.

$$\pi_0(t_j) = \pi_0(t_0) + [q_{00}\pi_0(t_0) + q_{10}\pi_1(t_0) + \pi_0(t_0)](t_j - t_0) + [\pi_0(t_0) - \pi_1(t_0)]\lambda_2 \quad (8.1)$$

Let $\Theta_j = (q_{01}, q_{10}, \lambda_0, \lambda_1, \pi_0(t_j))$. For each j, Θ_j is a function of Θ_0 because of (8.1). Generate the S_t process to get $\mathrm{E}_{\Theta_j}(e^{-\int_0^T \rho_s ds} X|\mathcal{F}_{t_j})$. These are all functions of Θ_0. Find Θ_0 for which these 5 expectations agree with the observed price of the option at the 5 points of time.

Another possibility of reducing the number of parameters to be estimated, which is same as the number of inversions is to assume that the θ process is in equilibrium when we start observing. Then we need to estimate (or invert for) 4 parameters $(q_{01}, q_{10}, \lambda_0, \lambda_1)$ since, following the notation of example 5.4(A) of Ross(1996):

$\mathrm{P}_{00}(t) = \frac{q_{10}}{q_{10}+q_{01}} + \frac{q_{01}}{q_{10}+q_{01}} e^{-(q_{10}+q_{01})t}$

$\mathrm{P}_{11}(t) = \frac{q_{01}}{q_{10}+q_{01}} + \frac{q_{10}}{q_{10}+q_{01}} e^{-(q_{10}+q_{01})t}$

The chain is irreducible and $T_{ii} \sim Exp(q_{10} + q_{01})$ that is non-lattice and finite mean. By Proposition 4.8.1 of Ross(1996):

$\pi_0(0) = \mathrm{P}(\theta_t = 0 | \theta_0 = j) = \frac{q_{10}}{q_{10}+q_{01}}$

8.5 Generalizations

According to Snyder(1973), the $\pi_i(t)$ process evolves as:

$$d\pi(t) = a(t)dt + b(t,1)dN_{1t} + b(t,2)dN_{2t}$$

where $a(t)$ and $b(t,i)$ are $\mathcal{F}_t$ adapted processes. Even if we do not want to specify the model for the λ process, it is still of that form. So the problems that we mentioned in section 7.2 remain, whatever be the model for the evolution of λ_t.

Let us see the form of the posterior in the general case. Let $c_t(v|N_{t_0,t})$ be the posterior characteristic function for λ_t given an observed path.

$$\begin{aligned} dc_t(v|N_{t_0,t}) &= \mathrm{E}(\exp(iv\lambda_t)\Psi_t(v|N_{t_0,t},\lambda_t)|N_{t_0,t})dt \\ &\quad -\mathrm{E}(\exp(iv\lambda_t)g(N_t)(\lambda_t - \hat{\lambda}_t)|N_{t_0,t}|)dt \\ &\quad +\mathrm{E}(\exp(iv\lambda_t)(\lambda_{N_t,N_{t+dt}} - \lambda^*_{N_t,N_{t+dt}})|N_{t_0,t})\lambda^{*-1}_{N_t,N_{t+dt}}d\xi_t \end{aligned}$$

$$\begin{aligned} \text{where } \lambda_{N_t,\zeta_t} &= \begin{cases} g(N_t)\lambda_t p_{\lambda_t,N_t} & \text{if} \quad \zeta_t = N_t + 1 \\ g(N_t)\lambda_t(1 - p_{\lambda_t,N_t}) & \text{if} \quad \zeta_t = N_t - 1 \\ 0 & \text{o. w.} \end{cases} \\ \lambda^*_{N_t,\zeta_t} &= \mathrm{E}(\lambda_{N_t,\zeta_t}(t, N_{t_0,t}, \lambda_t)|N_{t_0,t}) \\ \Psi_t(v|N_{t_0,t},\lambda_t) &= \mathrm{E}(exp(iv\Delta\lambda_t)|N_{t_0,t},\lambda_t) \end{aligned}$$

For example, let $d\lambda_t = f_t(\lambda_t)dt + G_t(\lambda_t)dW_t$, where W_t is Brownian motion. In this case $\Psi_t(v|N_{t_0,t},\lambda_t) = ivf_t(\lambda_t) - \frac{1}{2}v^2G_t^2(\lambda_t)$.

As long as the observed process is Markov jump process, the second and third terms are same as in the two state Markov case and hence yield the same function on taking inverse Fourier transform. The first term, on taking inverse Fourier transform yields L, the Kolmogorov-Fokker-Plank differential operator associated with λ.

$$\begin{aligned} d\pi_t &= L + \int_0^t \pi_j(u)(\bar{\lambda} - \lambda_j) g(S(u)) du \\ &+ \sum_{0<u<t} \pi_j(u-) \left(\frac{\lambda_j p_{\lambda_j}(S_u - \to S_u)}{\sum_i \pi_i(u) \lambda_i p_{\lambda_i}(S_u - \to S_u)} - 1 \right) \end{aligned}$$

Another possible direction is to consider jumps of size > 1. But then we no longer have the distribution of jump size from simple martingale considerations. We have to either assume the jump distribution, or estimate it, or impose some optimization criterion to get a unique price. Also, if the jump magnitude can take k values, we need $k-1$ market traded options and the stock to hedge an option.

Chapter 9

SIMULATIONS

9.1 Comparing the bounds on option price under various models

The computed minimum and maximum prices of a CALL option for several strike(K) prices under the continuous time jump model and under the discrete time approximation are plotted against the strike prices in Figure 9.1. The exact values are given in Table 9.2. The values of the parameters for the simulation are presented in Table 9.1. For the plot, in each case the Black Scholes price is subtracted from the prices. This makes the points more presentable on the same plot, as well as provides comparison with the Black-Scholes model. It is observed that in each case the minimum-maximum spread is quite small, which makes these intervals meaningful and convenient to use. The prices are substantially, though not drastically different from the Black-Scholes prices($1-$1.5). The discrete time approximations, which are

Table 9.1: The values of parameters

Stock price at time 0	$S_0 = \$100$
Expiration Time	$T = 90 * 24 * 60$
Rate of interest	$\rho = \ln(1.05)/(365 * 24 * 60)$
Volatility	$\sigma^2 = 1.0 * 10^{-8}$
Tick size	$1
Time unit for discrete time	$1/m$

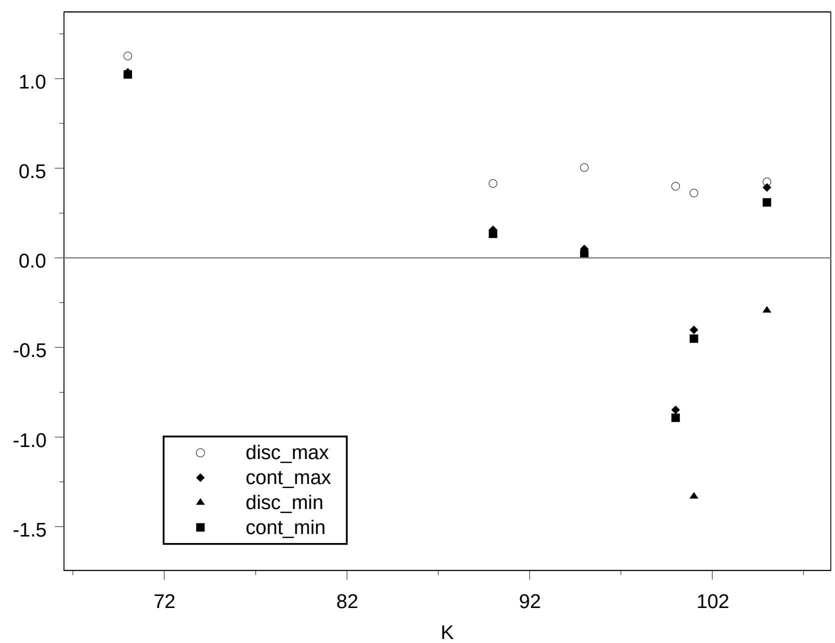

Figure 9.1: Prices under Different Models

Table 9.2: Prices of CALL option

	$K = 70$	$K = 90$	$K = 95$	$K = 100$	$K = 101$	$K = 105$
Black Scholes price	30.1869	11.0768	6.1912	2.1044	1.5409	0.2934
Discrete time combined max $m = 1$	31.3144	11.4955	6.7002	2.5158	1.9143	0.7254
Discrete time combined max $m \to \infty$	31.3131	11.4919	6.6947	2.5039	1.9024	0.7178
Continuous time exact max,$lim = 2$)	31.2226	11.2337	6.2410	1.2559	1.1387	0.6861
Discrete time combined min $m = 1$	31.2103	11.2103	6.2103	1.2103	0.2103	0.0000
Discrete time combined min $m \to \infty$	31.2103	11.2103	6.2103	1.2103	0.2103	0.0000
Continuous time exact min, $lim = 2$	31.2105	11.2111	6.2112	1.2114	1.0898	0.6035

computationally much faster, work reasonably well if the options are not near-the-money. Another notable aspect is the shape of the curve, which suggests that this model can perhaps counteract the implied volatility smile. This is because, under the same assumed volatility, near the money options have prices lower than Black-Scholes and in-the-money and out-of-money options have prices higher than Black-Scholes.

9.2 Maximum value the stock price can attain

While searching for the maximum option price, for computational purposes, we have to restrict the stock price to an interval. The lower limit of this interval is zero, because the stock price cannot be negative. There is, however, no natural upper limit. So we do a simulation study to find out how the bounds on option prices behave as we increase the limit on the maximum value of the stock price. Let us denote by *lim* this maximum value relative to the present stock price. That is, we shall consider $\{\sup_{0 \le t \le T} S_t \le lim \times S_0\}$. With $S_0 = \$15$, tick size =\$1/8 and strike $K = \$12$, Table 9.3 presents $\mathrm{P}(\xi_T = m)$ and *max* $\mathrm{E}[(S_T - K)_+ | \xi_T = m]$ for various

Table 9.3: Maximum

ξ_T	prob	1	1.1	1.2	1.5	2	10	20
0	0.855970	3	3	3	3	3	3	3
1	0.132394	–	4.1917	4.1917	4.1917	4.1917	4.1917	4.1917
2	0.011074	–	–	5.3522	5.3522	5.3522	5.3522	
3	0.000663	–	–	–	6.5841	6.5841	6.5841	
4	0.000032	–	–	–	7.8636	7.8947	8.3698	
E()	1.000133	–	–	–	3.186753	3.186754	3.186769	

Table 9.4: Minimum

ξ_T	prob	1	1.1	1.2	1.5	2	10	20
0	0.855970	3	3	3	3	3	3	3
1	0.132394	–	4.1603	4.1603	4.1603	4.1603	4.1603	4.1603
2	0.011074	–	–	5.3207	5.3207	5.3207	5.3207	
3	0.000663	–	–	–	6.5527	6.5526	6.5526	
4	0.000032	–	–	–	7.8014	7.8014	7.8014	
E()	1000133	–	–	–	3.182224	3.182224	3.182224	

values of m and lim. Table 9.4 does the same for the minimum. Recall that ξ_t is the number of jumps in time 0 to t. It is shown theoretically in Appendix E.1 that probability of $lim > 20$ is 0.05. It is seen here computationally that there is very little difference in price of the option between lim=20 and lim=1.5. Also, for each ξ_T we can compute the minimum and maximum for increasing values of lim and stop when there is very little increase.

If lim is too small, no possible three-tuple exists between 0 and $lim \times N_0$ so that the jump probabilities are non-negative. So some of the values are unavailable. Once there are possible three-tuples, increasing lim further doesn't change the minimum and maximum option price almost at all.

9.3 Robustness

The condition of convergence to Black -Scholes introduces constraints on the moments of the jump distribution. It might be of interest to studying how much the results are

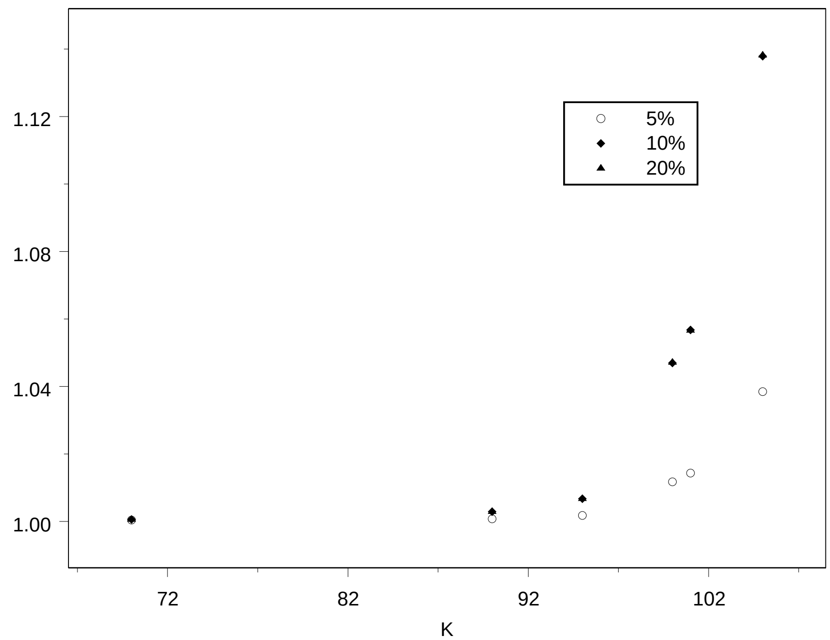

Figure 9.2: Plot for Robustness Study

affected if we relax these constraints slightly. The first moment condition is necessary for martingale properties. So we keep it unaltered. We relax the second moment conditions by 5% and compute the maximum under these new conditions. What this means is instead of considering distributions that have second moment exactly equal to N_t, we consider those with second moment between $0.95N_t$ and $1.05N_t$. We do the same thing at 10% and 20%. In Figure 9.2 we present the ratio of the maximum under the relaxed set of constraints to the maximum under the original constraints for various strike values(K) when the present stock price is \$100 and time to maturity is 3 months. It is observed that in-the-money CALL options are very robust. Also for all the options considered, the changes are almost same for 10% and 20% which suggests that the changes stabilize after sometime.

Chapter 10

REAL DATA APPLICATIONS

10.1 Description

Data on stock price and option price was obtained from the option-metrics database for three stocks: Ford, IBM and ABMD. The stock data is transaction by transaction. The format of the raw stock data is: SYMBOL, DATE, TIME, PRICE, SIZE, G127, CORR, COND, EX. After filtering for after hour and international market trading, the data is on tradings in NASDAQ regular hours. The prices are divided by the tick size to obtain integers.

The option data is daily best bid and ask prices. We preprocess the data to remove volume zero and symbols not equl to F,IBM or ABMD. For example, we do not consider the roots XFO, YFY, FOD and YOD which are on Ford stocks after a merger which pays $20 per stock + 1 stock of the new company. The option data has 'date, Call/Put, expiration, best-bid, best-ask, strike'.

We shall use the data for the first day of the month as training sample and for the rest of the days as test sample. Table 10.1 summarizes the average number of trades per day for each of the stocks and the dates for the training and test samples. In figures 10.1 to 10.8 we present the plots of paths of the 3 stocks. We estimate the risk-neutral parameters by inverting option prices in the training sample and use these estimates to predict prices of options in the test sample. We also obtain historical estimates of parameters from one year's data. The 3 stocks provide some variety. The

Table 10.1: Description of data

Stock	Trades per day	Dates for sample	
		Training	Test
F	1675	Dec 4, 2002	Dec 5-Dec 31, 2002
ABMD	400	Feb 3, 2003	Feb 4-Feb 28, 2003
IBM	4270	June 3, 2002	June 4-June 30, 2002

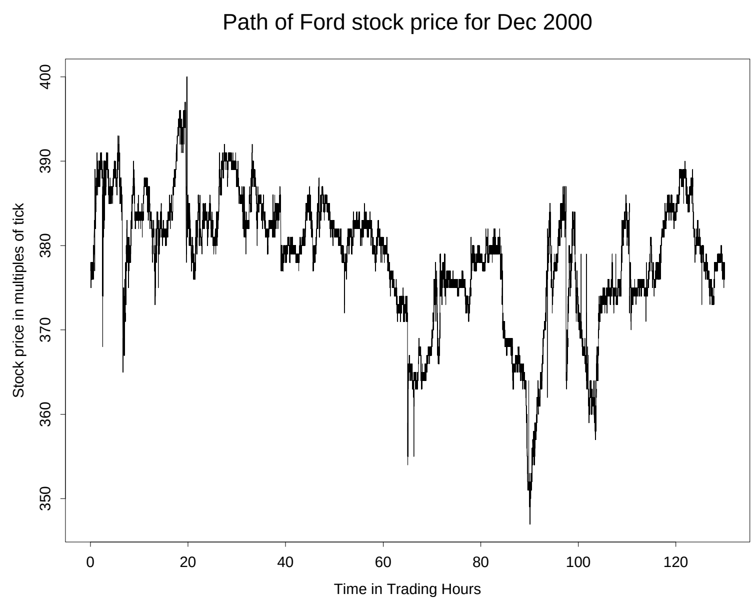

Figure 10.1: Ford Path 1

Ford stock is a little old when the tick size used to be \$1/16 while the others have tick size \$1/100. This should shed some light on how much effect the change in tick size has on the analysis. The ABMD data is much more thinly traded than the other two, as will be evident from the plots of paths of stock prices. So while the IBM data can be well approximated by a continuous path and it might still be alright for the Ford data, the continuity assumption for the ABMD data is definitely too far-fetched.

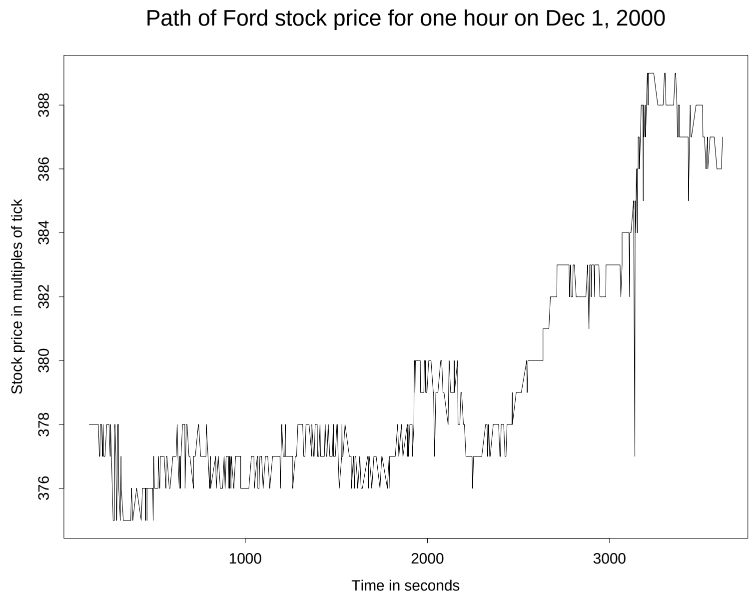

Figure 10.2: Ford Path 2

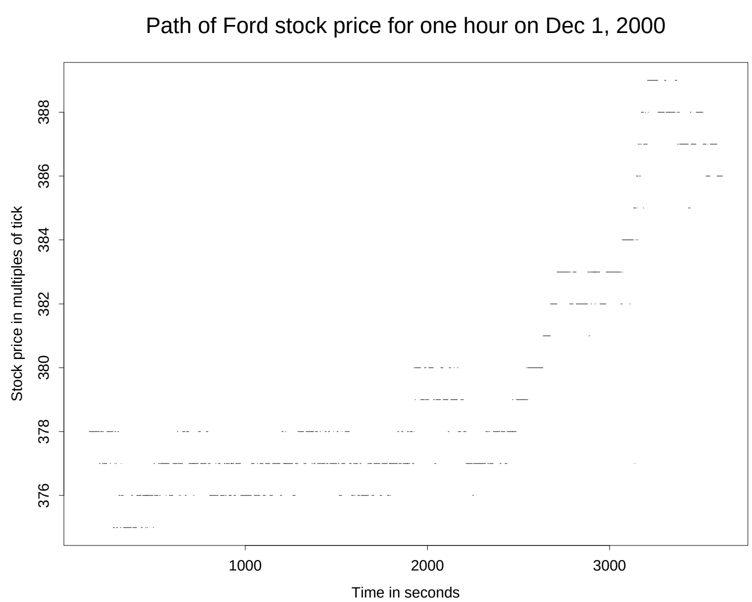

Figure 10.3: Ford Path 3

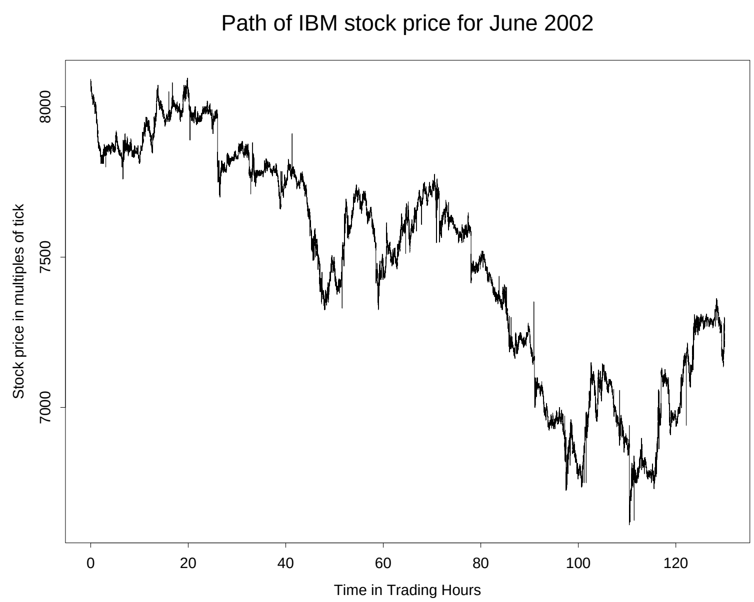

Figure 10.4: IBM Path 1

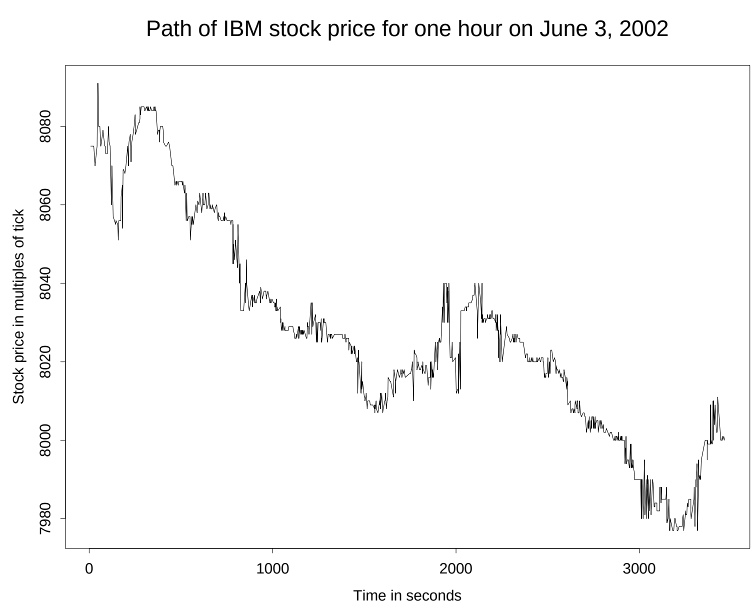

Figure 10.5: IBM Path 2

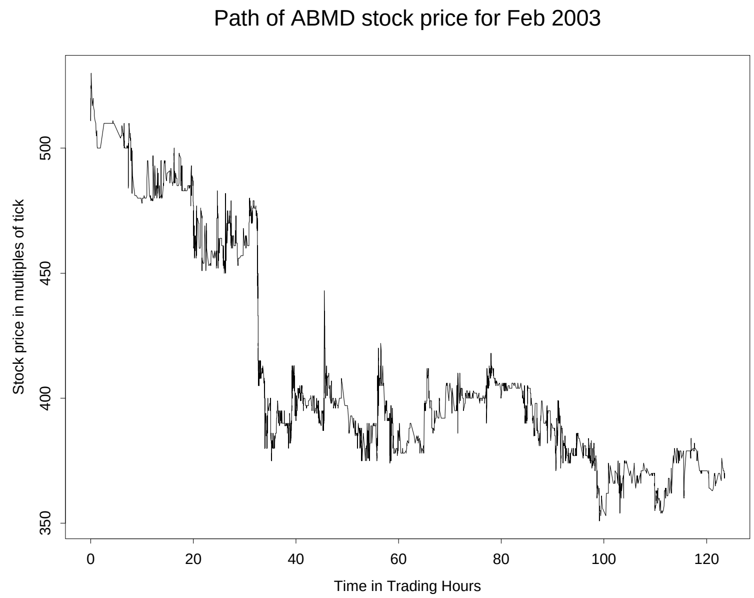

Figure 10.6: ABMD Path 1

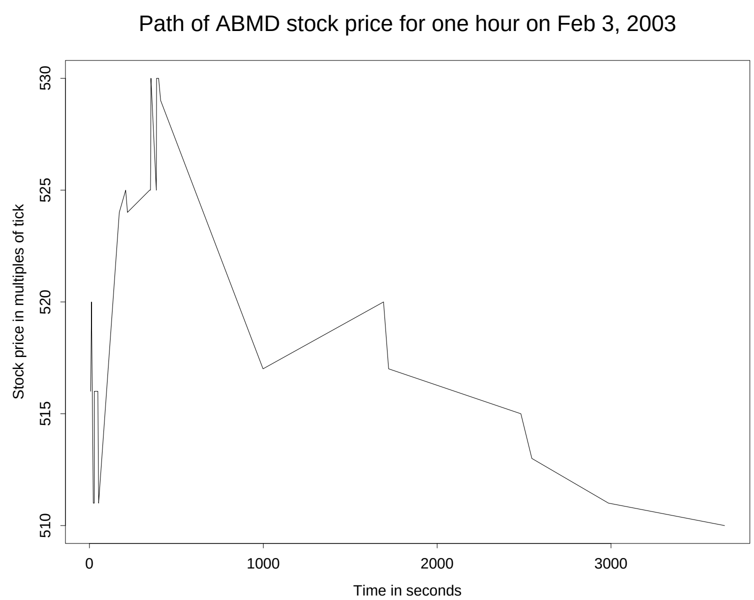

Figure 10.7: ABMD Path 2

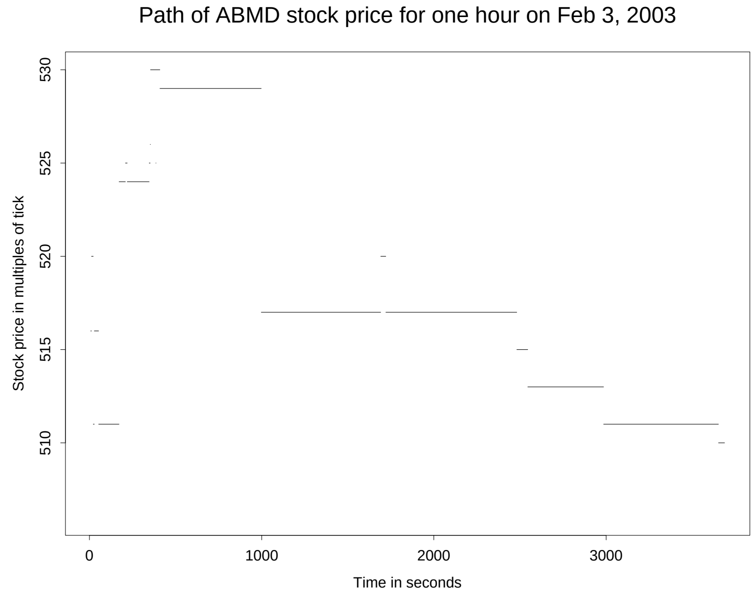

Figure 10.8: ABMD Path 3

Table 10.2: Historical Estimates

Symbol	Date Range	Intensity rate $\hat{\lambda}$	Jump $\hat{\nu}$	Black Scholes $\hat{\sigma}^2$
F	Dec 1,2001-Nov 30, 2002	3.43224e-03	3.07369e-01	1.72185e-09
IBM	June 1, 2001-May 31, 2002	4.65843e-04	8.08839e-02	5.64927e-11
ABMD	Feb 1, 2002- Jan 31, 2003	2.45297e-04	1.63711e-02	5.25181e-10

10.2 Historical Estimation

Table 10.2 gives historical estimates of parameters based on one year raw stock data. Black Scholes $\hat{\sigma^2} := \text{Var}[(\log(S_i/S_{i-1}))]/(\text{total trading time})$. The Jump estimates are the likelihood and quasi-likelihood estimates as described in section 3.3. All units are ticks and minutes.

10.3 General jumps

For the general jump model, we obtain the intervals for option prices from the model under various values of the intensity rate parameter. For the learning sample of

Table 10.3: Summary of Training Sample

Stock	Sample size	Range of option prices	Range of parameters	Average observed bid-ask spread
ABMD	12	0-160	1×10^{-9} to 10×10^{-9}	25
F	34	0-180	0.5×10^{-7} to 4.5×10^{-7}	2
IBM	92	0-3700	0.7×10^{-9} to 2×10^{-9}	10

ABMD, we have 12 options that are traded on Feb 3, 2003. For values of the intensity parameter between 1×10^{-9} to 10×10^{-9}, we obtain the intervals based on simulations of size 1000. Figure 10.9 plots the length of the predicted interval against the distance of the predicted interval from the observed interval, as obtained for various values of the parameter. The range of option prices is 0-160. The average observed bid-ask spread is 25. The unit is 1c. The distance is measured as the average distance between the midpoint of bid and ask and the point on the predicted interval that is closest to the observed interval. Similar plots for the IBM and Ford data are in figures 10.11(a) and 10.13. Table 10.3 summarizes the various characteristics of the training samples for the 3 stocks. All numbers are in multiples of tick. The number of replications for the simulations are always 1000.

We choose one of the parameter values and use that to predict the range of option prices for the test samples. The various characteristics of the test samples for the 3 stocks are summarized in table 10.4. These include the average length of the predicted interval and the distance of the predicted interval from the observed interval, as obtained from the analysis. We present the plots showing the midpoints of observed bid-ask intervals and the corresponding predicted intervals for the test samples in figures 10.10, 10.12 and 10.14. For the IBM data we also present a similar plot for the training sample in figure 10.11(b).

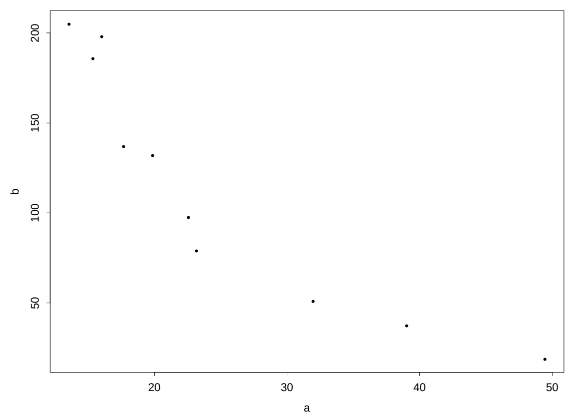

Figure 10.9: ABMD Training: Length of predicted interval vs distance of predicted interval from observed interval

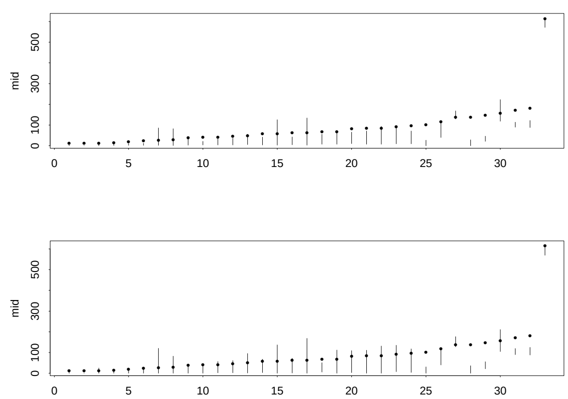

Figure 10.10: ABMD Prediction: Predicted intervals and bid-ask midpoint

Table 10.4: Summary of test sample

Stock	Sample size	Number of replications	Parameter	Range of option prices	Average length	Average distance
ABMD	33	1000	3×10^{-9}	0-550	47.18	18.59
ABMD	33	10000	3×10^{-9}	0-550	66.56	13.06
F	578	1000	2×10^{-7}	0-256	17.14	4.33
IBM	1823	1000	1.4×10^{-9}	0-8000	273.17	55.68

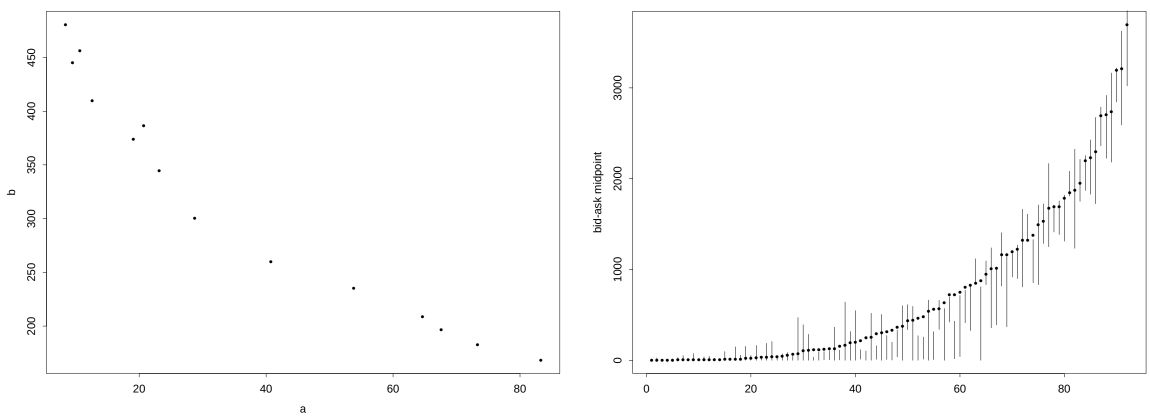

Figure 10.11: IBM Training (a)Length of predicted interval vs distance of predicted interval from observed interval (b)Predicted intervals and bid-ask midpoint

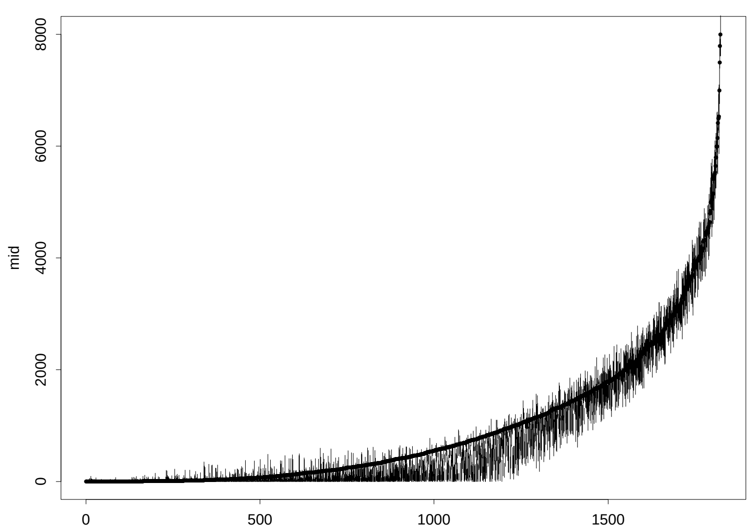

Figure 10.12: IBM Prediction: Predicted intervals and bid-ask midpoint

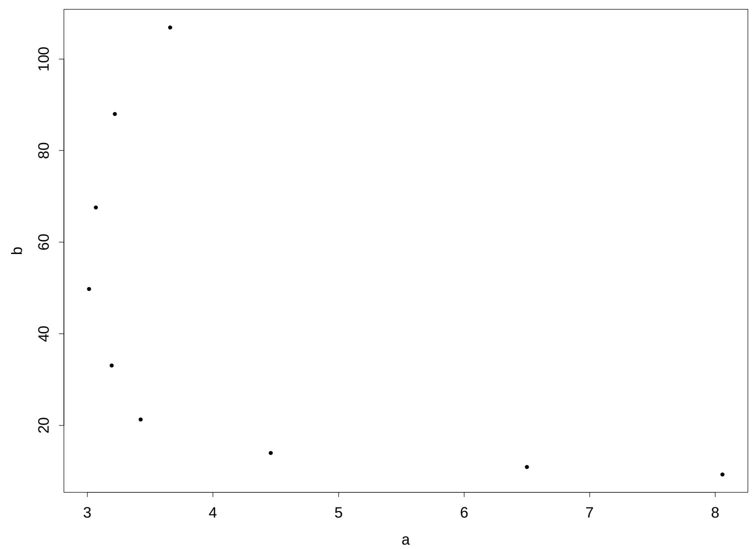

Figure 10.13: Ford Training: Length of predicted interval vs distance of predicted interval from observed interval

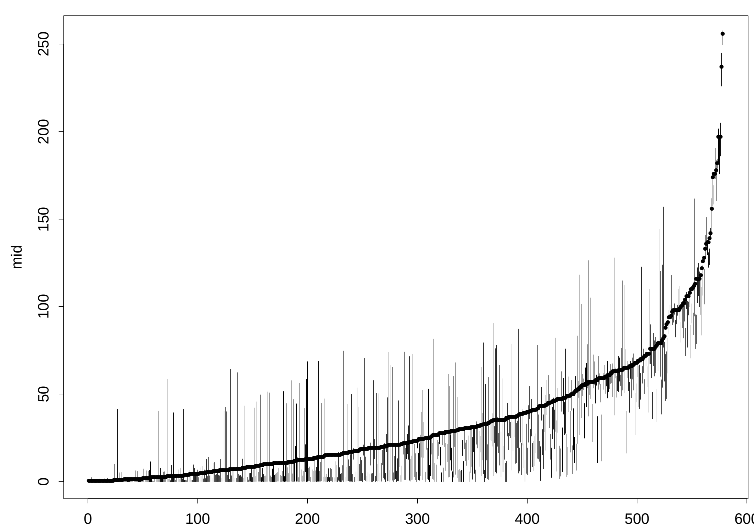

Figure 10.14: Ford Prediction: Predicted intervals and bid-ask midpoint

10.4 Birth-Death process

10.4.1 Constant Intensity Rate

The predicted option price from the model is computed at seven points during the day at the first times that a trade takes place on the stock at every hour. The minimum, maximum and average of these 7 predicted values will be denoted by min, max and avg respectively. The observed interval of daily best bid and ask values is (bid, ask) and mid:=(ask+bid)/2. For a discussion on the issue of using the midpoint when data is recorded as discrete bid-ask quotes, refer to Jones et al(1994) and Hasbrouck(1999).

The performance of a method is good if the predicted interval (min,max) is close to the observed interval (bid,ask). We use the following measures of distance between 2 intervals to summarize the results:

a=Average of [(bid-max)$_+$+(min-ask)$_+$]

b=Average of [(mid-max)$_+$+(min-mid)$_+$]

c =Root mean square[avg-mid]

Table 10.5 presents the parameter, the average length of predicted interval and the 3 distance measures a, b, c for the 2 methods and various data-sets. Here, the parameter is the volatility for the Black-Scholes model and the intensity rate for the birth-death model. It is observed that the Black Scholes model with constant volatility and Birth Death model with constant intensity give comparable results. In both of these cases once we fix the parameter, we are considering one fixed model. We get the intervals as the maximum and minimum of the option prices as the stock price varies over the day. So these are not any prediction intervals, but just arise because of intra-day variation in the option price. With this in mind, these are like point estimates, whereas the intervals obtained in section 10.3 are like interval estimates

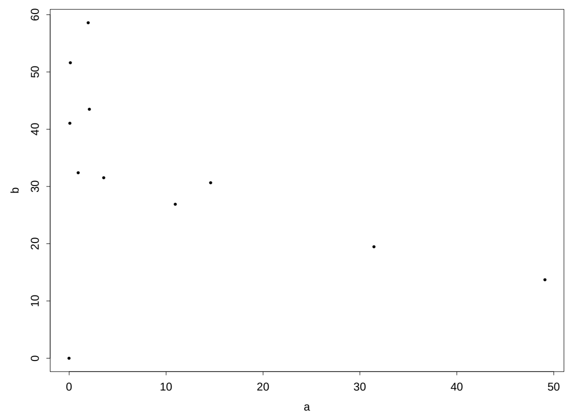

Figure 10.15: ABMD Prediction for birth death model: Length of predicted interval vs distance of predicted interval from observed interval

because these we are considering all models(that is all possible jump size distribution) with a particular intensity rate. In the birth-death the jump size distribution is fixed. It is ± 1 with probability of positive jump fixed from martingale considerations once the intensity is fixed. Thus if we compare Figure 10.9 with Figure 10.15, the intervals in the former are much larger. In figures 10.4.1 to 10.4.1 we present the error (observed-expected) using mid-points of the intervals in pricing of options against strikes conditional on number of days to expiration. It is seen that the CALL options are priced higher by the model and the PUTs are priced lower. Also we observe the phenomenon of smile that is common to all option pricing models. From the test sample data it is noticed that options that are in the money have higher variability of pricing errors. This is also a common phenomenon of many option pricing models, the cause for which is attributed to higher trading of in the money options than out of the money options.

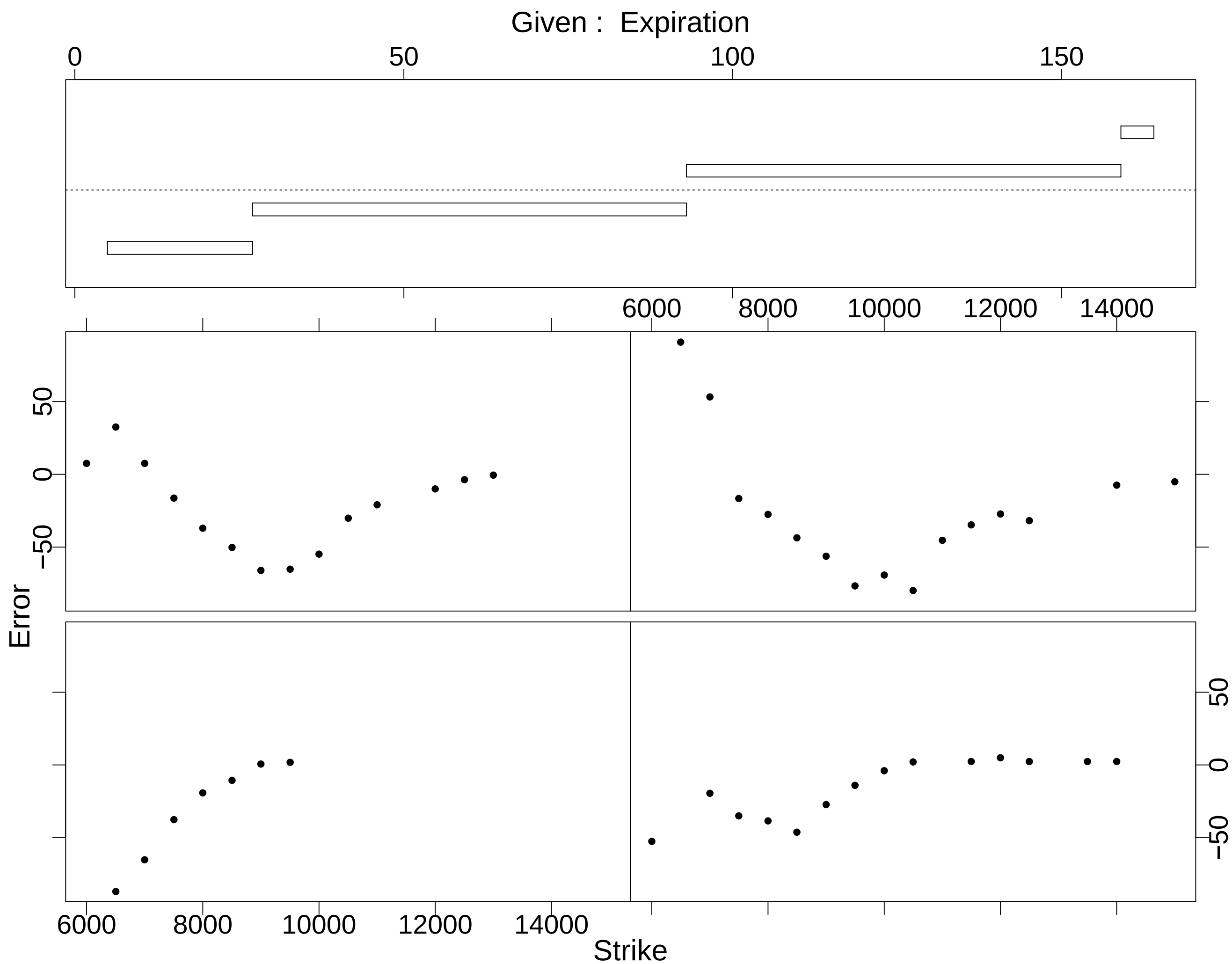

Figure 10.16: Error in CALL price for training sample of IBM data

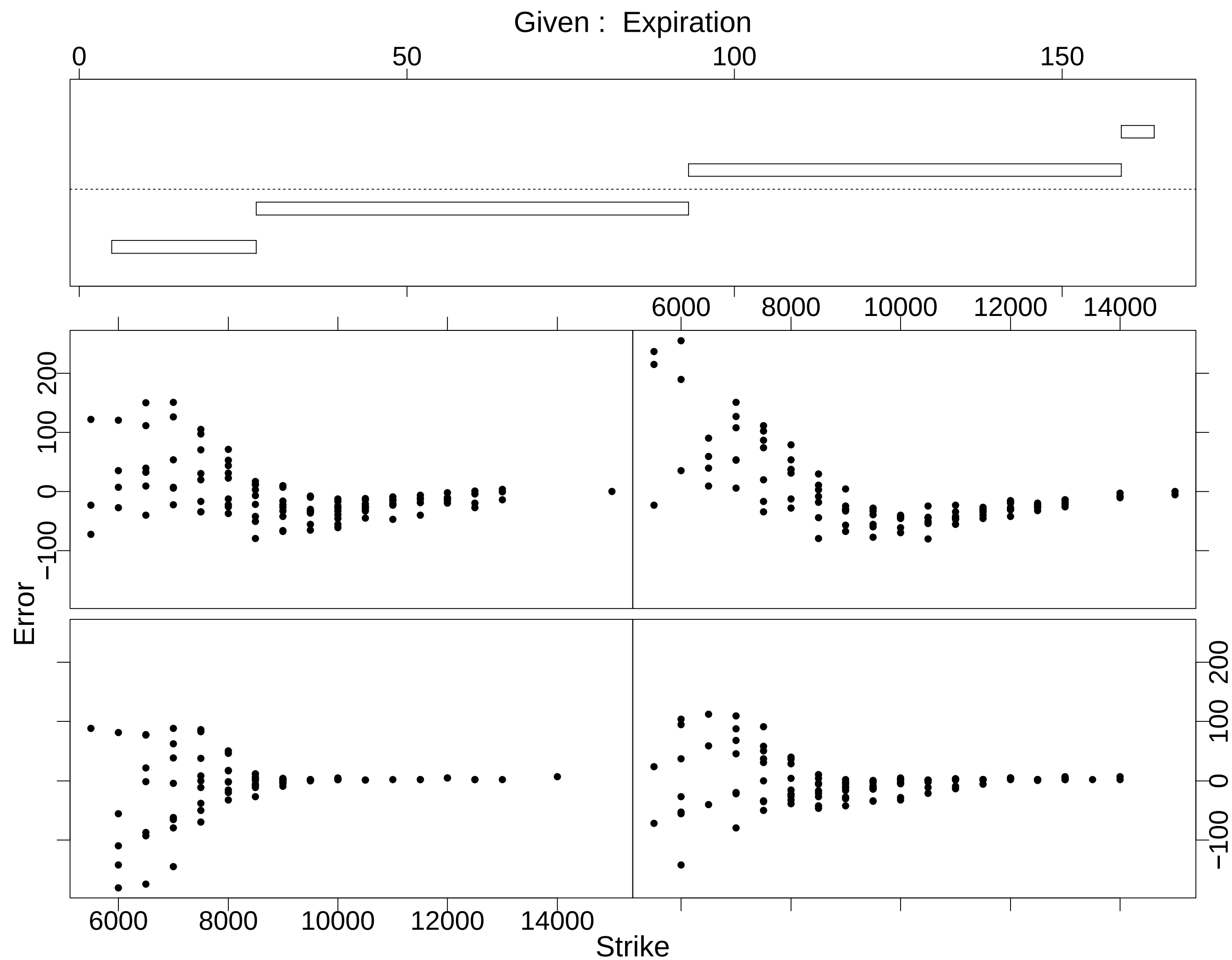

Figure 10.17: Error in CALL price for test sample of IBM data

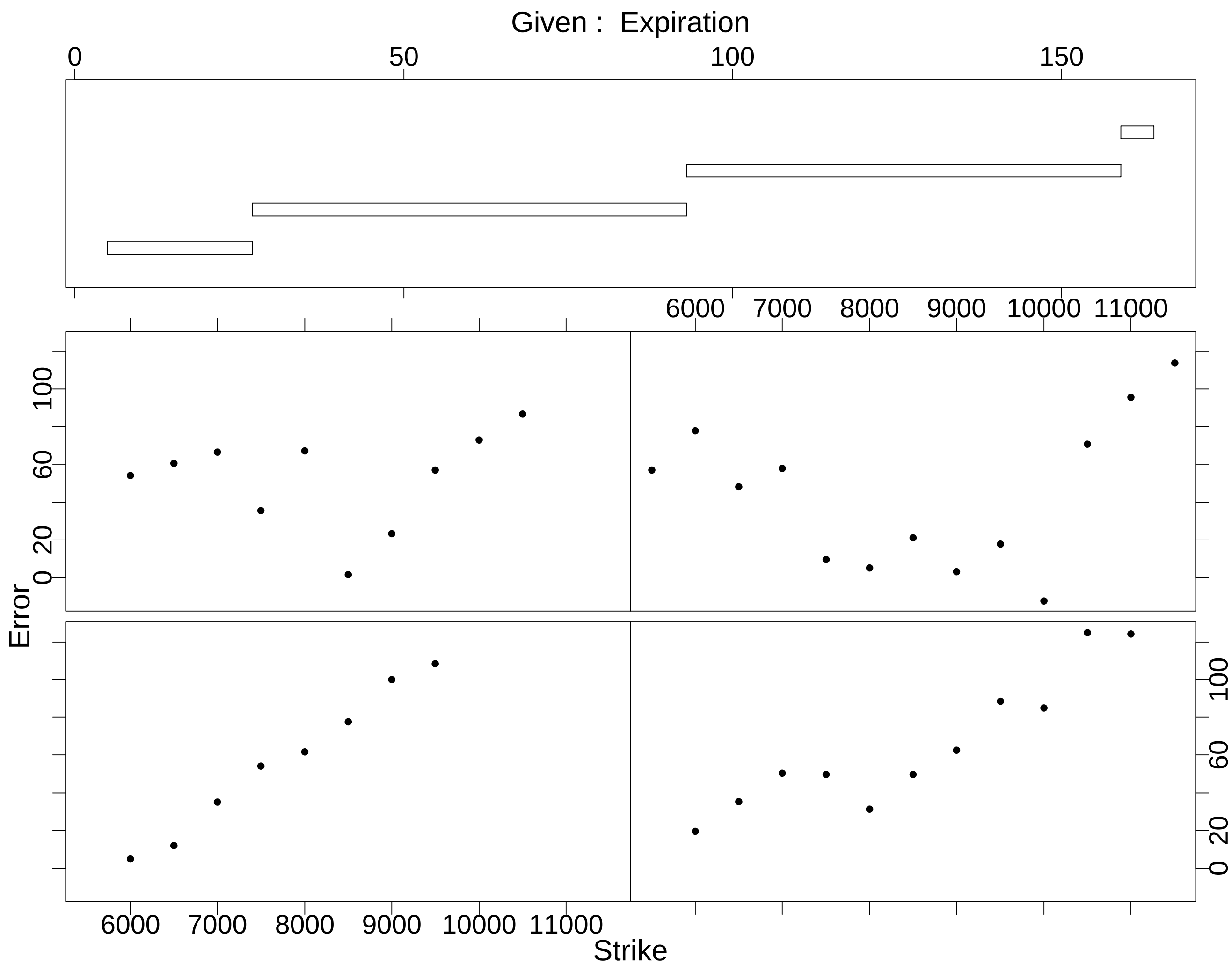

Figure 10.18: Error in PUT price for training sample of IBM data

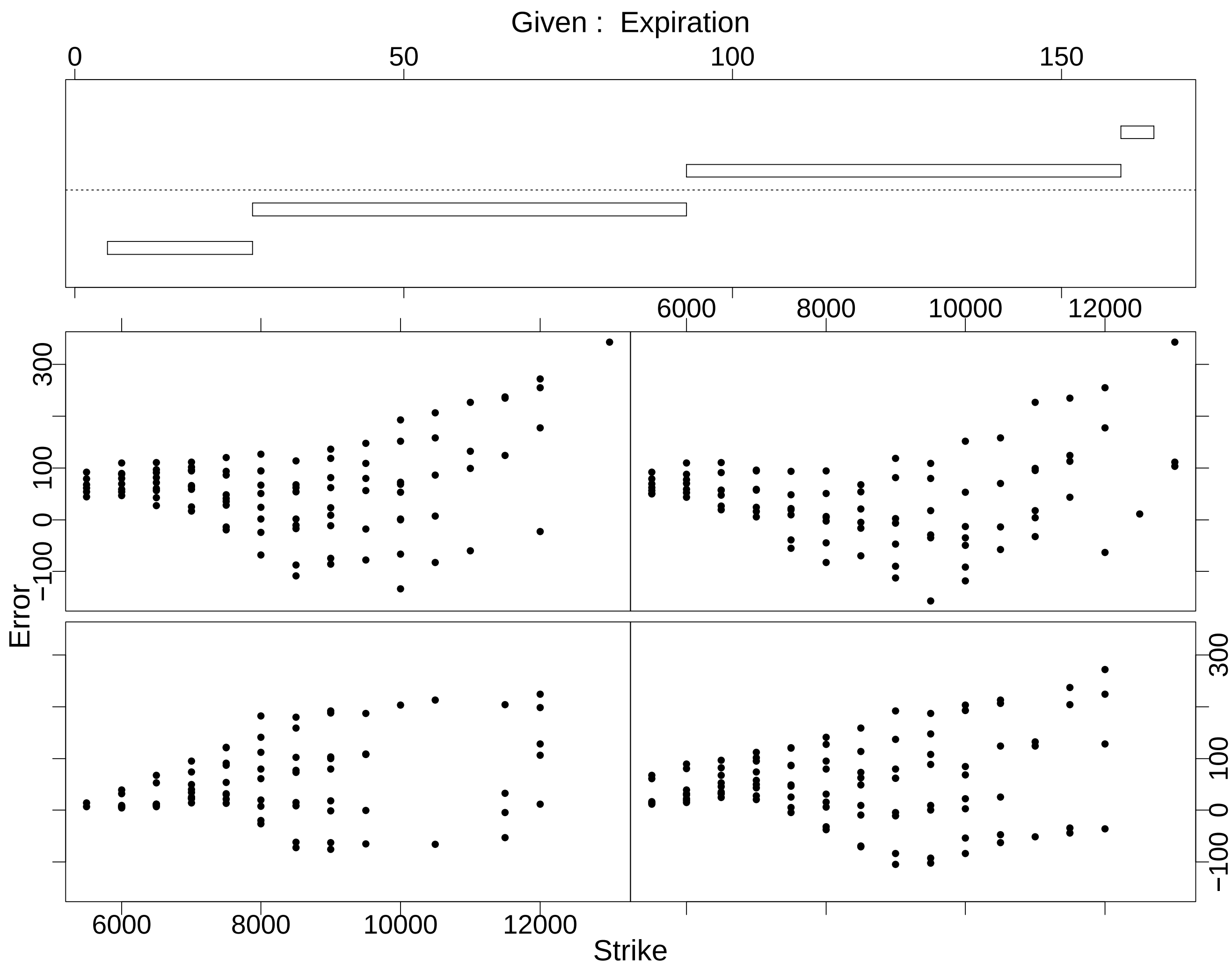

Figure 10.19: Error in PUT price for test sample of IBM data

Table 10.5: Results for constant intensity rate

Model	Sample	Data	Parameter	Length	a	b	c
BS	Training	IBM	8.0e-07	99.1806	8.5836	10.6033	46.7688
BS	Test	IBM	8.0e-07	52.1990	14.8347	19.4810	61.4072
BD	Training	IBM	1.1e-06	124.4025	12.1834	10.112	52.7742
BD	Test	IBM	1.1e-06	81.0485	16.5362	20.5029	70.6355
BS	Training	ABMD	6.5e-06	9.6373	0.0000	1.0318	4.7487
BS	Test	ABMD	6.5e-06	12.7685	0.6439	3.8731	9.3419
BD	Training	ABMD	7.5e-06	17.5405	0.0000	0.9736	5.8723
BD	Test	ABMD	7.5e-06	19.4657	0.1810	2.4301	9.9976
BS	Training	F	9.0e-07	6.0966	0.0873	0.2971	1.5191
BS	Test	F	9.0e-07	5.9190	0.3477	0.6092	3.2799
BD	Training	F	1.3e-06	6.7910	0.6278	0.2259	1.7516
BD	Test	F	1.3e-06	6.5784	1.3237	0.8868	4.0336

10.4.2 Stochastic Intensity Rate

For stochastic intensity rate, we have to find the price of options given 5 parameters: the two intensity parameters λ_0, λ_1, the transition matrix of the intensity process that is determined by q_{01}, q_{10}, and π the probability that the intensity process is in state λ_0 at time 0. The objective is to find the parameter set that minimizes the root mean square error between the bid-ask-midpoint and the daily average of the predicted option price, for all options in the training sample. We followed a diagonally scaled steepest descent algorithm with central difference approximation to the differential. The starting values of λ_0, λ_1 are taken to be equal to the value of the estimator $\hat{\lambda}$ obtained in the constant intensity model. The starting values of q_{01}, q_{10} are obtained by a hidden Markov model approach using an iterative method that has 2 steps. Let the underlying state process be η_t, that is the intensity is λ_i when η_t is i. In one step, the MLE of the parameters is obtained given the η_t process. For details on the method for obtaining the MLE in this setting, refer to pp 172 of Elliott(1995). Given the latent process the MLE's are obtained from the following equations:

$$\hat{q}_{ij} = \frac{\text{Number of times latent process jumps from state } i \text{ to state } j}{\text{Time spent by latent process in state } i}$$

$$\sum_{j \in \text{State space}} \frac{\mathcal{A}^m_{j,j+1}}{\lambda_m + \frac{r}{j}} + \frac{\mathcal{A}^m_{j,j-1}}{\lambda_m - \frac{r}{j}} - j^2 \mathcal{T}_t^{j,m} = 0$$

where $\mathcal{A}^m_{i,j}$ = Number of times observed process jumps from state i to state j when latent process is in state m, and $\mathcal{T}_t^{j,m}$= Time during which latent process is in state m and observed process is in state j.

In the next step, for each t when there is a jump in the stock price, we assign η_t to that i which maximizes the probability of an event. When this method converges, we do a finite search on the parameter π. Then we perform the minimization algorithm to get the risk neutral parameters.

For the ABMD and Ford datasets, the RMSE of prediction obtained from the constant intensity method is less than the bid-ask spread. This means that the constant intensity model attains the lower bound on the possible quadratic calibration error as referred to the intrinsic error in Cont(2004). Since the data is observed with an error, that is only as bid-ask quotes, we cannot hope to achieve a level of error less than the precision of the observed data. However for the IBM data there is scope for improvement.

Table 10.6 shows the values of λ_0, λ_1 and the RMS error for the evolution of the algorithm. The initial values of q_{01} , q_{10} and π are 8.64e-02, 1.2126 and 0.5 and they do not change in the significant digits over the evolution of the algorithm.

The algorithm converges with the value of RMSE equal to 29.6799. It is seen that we do achieve substantial improvement in the RMSE by using the stochastic intensity rate model over the birth death model with constant intensity (RMSE=52.7729) or

Table 10.6: Evolution of algorithm

λ_0	λ_1	RMSE
1.100000e-06	1.100000e-06	52.7742
1.065162e-06	1.594270e-07	39.5022
1.043488e-06	8.320846e-08	38.2857
1.042039e-06	8.326884e-08	29.6799

Black-Scholes model with constant volatility (RMSE=46.7688). However, the bias and smile that were observed in the constant volatility model still remain for the stochastic volatility model. Also, this is an ill-posed problem in the sense that the solutions are not unique and depend on the starting values of the parameters. There is still scope for improvement since the best RMSE we can hope to attain is the daily bid-ask spread which is 5 in this case.

Appendix A

Proof of Proposition 3.2.0.1

Let X be a process driven by the stochastic differential equation $dX_t = (\rho_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t$, where W_t is standard Weiner process. In particular X is a process with independent increments and characteristics $(\int_0^T (\rho_t - \sigma_t^2/2)dt, \int_0^T \sigma_t^2 dt, 0)$.
Let $X_t^{(n)} = \log(N_t^{(n)}/N_0^{(n)})$. From (6.10) in Mykland(1994)

$$< X_t^{(n)}, X_t^{(n)} >_t = \int_0^T \mathrm{E}\left[(\Delta X_t^{(n)})^2 \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 dt \xrightarrow{\mathrm{P}} \int_0^T \sigma_t^2 dt \tag{A.1}$$

$$< X_t^{(n)} >_t = \int_0^T \mathrm{E}\left[(\Delta X_t^{(n)}) \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 dt \xrightarrow{\mathrm{P}} \int_0^T (\rho_t - \sigma_t^2/2) dt \tag{A.2}$$

(A.1), (A.2), assumption B1 and Theorem VIII.3.6 of Jacod and Shiryaev(2002) imply $X^{(n)} \xrightarrow{\mathcal{L}} X$. Since exp is a continuous function, $S^{(n)} = \exp(X^{(n)}) \xrightarrow{\mathcal{L}} \exp(X) =: S$. By Ito's formula,

$$dS_t = S_t[(\rho_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t] + \frac{1}{2} S_t \sigma_t^2 dt = S_t \rho_t dt + S_t \sigma_t dW_t$$

We shall now prove that if $\mathrm{E}[Y_t^{(n)} \mid \mathcal{F}_{t-}^{(n)}] = \rho_t/\sigma_t^2$, $\mathrm{E}[Y_t^{(n)2} \mid \mathcal{F}_{t-}^{(n)}] = N_{t-}^{(n)}$ and $\mid Y_t^{(n)} \mid \leq k N_{t-}^{(n)\delta}$ where $0 < k < 1$ and $\delta < 2/3$, then

$$\mathrm{E}\left[(\Delta X_t^{(n)}) \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 \xrightarrow{\mathrm{P}} \rho_t - \sigma_t^2$$
$$\mathrm{E}\left[(\Delta X_t^{(n)})^2 \middle| \mathcal{F}_{t-}\right] N_{t-}^{(n)} \sigma_t^2 \xrightarrow{\mathrm{P}} \sigma_t^2 du$$

$$\mathrm{E}\left[(\Delta X_t^{(n)})\middle|\mathcal{F}_{t-}\right] N_{t-}^{(n)}\sigma_t^2 = \mathrm{E}\left[\ln\left(1+\frac{Y_t}{N_{t-}}\right) N_{t-}\sigma_t^2\middle|\mathcal{F}_{t-}\right]$$

$$\begin{aligned}\left|\ln\left(1+\frac{Y_t}{N_{t-}}\right)-\left[\frac{Y_t}{N_{t-}}-\frac{1}{2}\frac{Y_t^2}{N_{t-}^2}\right]\right|N_{t-}\sigma_t^2 &\le \left|\sum_{j\ge 3}\frac{1}{j}(-1)^{j-1}\frac{Y_t^j}{N_{t-}^j}\right|N_{t-}\sigma_t^2\\ &\le \sum_{j\ge 3}\frac{1}{j}\frac{\mid Y_t^j\mid}{N_{t-}^{j-1}}\sigma_t^2\\ &\le \sigma_t^2\sum_{j\ge 3}\frac{k^j}{j}\frac{N_{t-}^{j\delta}}{N_{t-}^{j-1}}\end{aligned}$$

Since $j \ge 3$ and $\delta < 2/3$, $j-1-j\delta > 0$. Also, $N_{t-} \ge 1$. Hence the last expression is bounded above by $\sigma_t^2 \sum_{j\ge 3}\frac{k^j}{j}$ and goes to 0 a.s. as $n \to \infty$. Hence,

$$\mathrm{E}\left[\ln\left(1+\frac{Y_t}{N_{t-}}\right)N_{t-}\sigma_t^2\middle|\mathcal{F}_{t-}\right] \longrightarrow \mathrm{E}\left[\left(\frac{Y_t}{N_{t-}}-\frac{1}{2}\frac{Y_t^2}{N_{t-}^2}\right)N_{t-}\sigma_t^2\middle|\mathcal{F}_{t-}\right] = \rho_t - \sigma_t^2/2$$

The other convergence can be proved similarly by considering the Taylor expansion of $\ln(1+\frac{Y_t}{N_{t-}})^2$

Appendix B

Proof of Proposition 5.1.0.2

$$
\begin{aligned}
X_t^{(n)} &= ln(\frac{N_t^{(n)}}{n}) \\
X_t^{*(n)} &= X_t^{(n)} - \int_0^t [p_{u,N_u} \log(1+\frac{1}{N_u}) + (1-p_{u,N_u}) \log(1-\frac{1}{N_u})] N_u^2 \sigma_u^2 du - X_0^{(n)} \\
\text{where} \quad & p_{t,N_t} = \frac{1}{2}(1+\frac{\rho_t}{N_t \sigma_t^2})
\end{aligned}
$$

$$
\mid \Delta X_t^{*(n)} \mid = \log(1+\frac{Y_t}{N_t}) \leq \log(2) \quad \text{identically} \tag{B.1}
$$

Lemma B.0.2.1. $\forall t > 0, [p_{t,N_t} \log(1+\frac{1}{N_t}) + (1-p_{t,N_t}) \log(1-\frac{1}{N_t})] N_t^2 \sigma_t^2 \xrightarrow{\text{P}} \rho_t - \frac{\sigma_t^2}{2}$

Proof. $[p_{t,N_t} \log(1+\frac{1}{N_t}) + (1-p_{t,N_t}) \log(1-\frac{1}{N_t})] N_t^2 \sigma_t^2$
$= (2p_{t,N_t} - 1) N_t^2 \sigma_t^2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)N_t^{2k-1}} - N_t^2 \sigma_t^2 \sum_{k=1}^{\infty} \frac{1}{2kN_t^{2k}}$
$= \rho_t - \frac{\sigma_t^2}{2} + \rho_t \sum_{k=1}^{\infty} \frac{1}{(2k+1)N_t^{2k}} - \sigma_t^2 \sum_{k=1}^{\infty} \frac{1}{(2k+2)N_t^{2k}}$
$\xrightarrow{\text{P}} \rho_t - \frac{\sigma_t^2}{2}$ □

Lemma B.0.2.2. $X_t^{*(n)}$ *is a local martingale.*

Proof. Let $dN_{1t} = I_{Y_t=+1} dN_t$ and $dN_{2t} = -I_{Y_t=-1} dN_t$ with $N_{i0} = 0$. (N_{1t}, N_{2t}) is 2-dimensional point processes with $dN_t = dN_{1t} - dN_{2t}$ and intensity $(\lambda_{1t} := \sigma_t^2 p_{t,N_t} N_t^2, \lambda_{2t} := \sigma_t^2 (1-p_{t,N_t}) N_t^2)$. $Q_{it} = N_{it} - \int_0^t \lambda_{iu} N_u^2 du$ is the compensated point process associated with N_{it} and hence by thm T8 of Bremaud(1981), $X_t^{*(n)} = \int_0^t \log(1+\frac{1}{N_u}) dQ_{1u} + \int_0^u \log(1-\frac{1}{N_u}) dQ_{1u}$ is a local martingale if $\int_0^t \mid \log(1+\frac{1}{N_u}) \mid$

$N_u^2\lambda_u du$ and $\int_0^t \mid \log(1-\frac{1}{N_u}) \mid N_u^2\lambda_u du$ are finite a.s. This is true by (B.1) and assumption H2 □

Lemma B.0.2.3. $[X_t^{*(n)}, X_t^{*(n)}]_t \xrightarrow{\text{P}} \int_0^t \sigma_u^2 du \quad \forall t \in D$ dense

Proof. Using the same technique as in lemma B.0.2.2, we can show that: $\int_0^t \{log(1+\frac{1}{N_u})\}^2 dQ_{1u} + \int_0^t \{\log(1-\frac{1}{N_u})\}^2 dQ_{1u}$ is a local martingale if $\int_0^t \mid \log(1+\frac{1}{N_u}) \mid^2 N_u^2\sigma_u^2 du$ and $\int_0^t \mid \log(1-\frac{1}{N_u}) \mid^2 N_u^2\sigma_u^2 du$ are finite a.s. This is true by (B.1) and assumption H2

Hence,

$$\frac{d}{dt} < X^{*(n)}, X^{*(n)} >_t = [p_{t,N_t}\{\log(1+\frac{1}{N_t})\}^2 + (1-p_{t,N_t})\{\log(1-\frac{1}{N_t})\}^2]N_t^2\sigma_t^2$$

An expansion similar to the one in Lemma B.0.2.1 shows that this quantity converges in probability to σ_t^2 for all $t > 0$.

$$\text{Let} M_t^{(n)} = [X_t^{*(n)}, X_t^{*(n)}] - < X^{*(n)}, X^{*(n)} >_t$$

$$\text{if Assumption } H1 \text{ holds, } [M^{(n)}, M^{(n)}]_t = \sum_{\tau_i \leq t} \mid \Delta X_{\tau_i}^{(n)} \mid^4 \xrightarrow{\text{P}} 0$$

and the result follows □

Lemma B.0.2.2, Lemma B.0.2.3, (B.1) and Thm VIII.3.11 of Jacod and Shiryaev (2002) imply $X^{*(n)}$ converges in law to a continuous Gaussian Martingale with characteristics $(0, \int_0^t \sigma_u^2 du, 0)$. This and Lemma B.0.2.1 imply $X^{(n)}$ converges in law to X, a continuous Gaussian Martingale with characteristics $(\int_0^t (\rho_u - \frac{1}{2}\sigma_u^2)du, \int_0^t \sigma_u^2 du, 0)$.

Appendix C

Edgeworth Expansion

We shall use the following results from Mykland (1995):

Theorem C.0.2.1. *Let* $(\ell_t^{n,i})_{0\le t\le T_n}$ *be a triangular array of zero mean cadlag martingales. Assume conditions (I1)-(I3) and that* ϱ^{ij} *is well defined. Then*

$$\mathrm{P}(c_n^{-1/2}\ell_{T_n}^{n,\cdot} \le x) = \Phi(x,\kappa) + r_n J(.) + o_2(r_n) \tag{C.1}$$

This is Theorem 1 of Mykland (1995). The conditions are:

(I1) For each i, there are $\underline{k}_i, \bar{k}_i, \quad \underline{k}_i < \kappa^{i,j} < \bar{k}_i$ so that

$$r_n^{-1}\left(\frac{(\ell^{n,i},\ell^{n,i})_{T_n}}{c_n} - \kappa^{i,i}\right)\mathbf{I}(\underline{k}_i \le c_n^{-1}(\ell^{n,i},\ell^{n,i})_{T_n} \le \bar{k}_i)$$

is uniformly integrable

(I2) For the same $\underline{k}_i, \bar{k}_i$,

$$\mathrm{P}(\underline{k}_i \le c_n^{-1}(\ell^{n,i},\ell^{n,i})_{T_n} \le \bar{k}_i) = 1 - o(r_n)$$

for each i

(I3) For each i,

$$\mathrm{E}[\ell^{n,i},\ell^{n,i},\ell^{n,i},\ell^{n,i}]_{T_n} = O(c_n^2 r_n^2)$$

$o_2(.)$ in (C.1) denotes test function convergence and should be taken to mean that

$$\mathrm{E}g(c_n^{-1/2}\ell^{n,\cdot}_{T_n}) = \mathrm{E}g(N(0,\kappa^{i,j})) + r_n J(g) + o(r_n) \tag{C.2}$$

holds uniformly in the class $\mathcal{C}$ described in section 5.2 of functions g. One can take

$$\varrho^{i,j} = \mathrm{E}_{\text{as}}\left(r_n^{-1}\left(\frac{(\ell^{n,i},\ell^{n,i})_{T_n}}{c_n} - \kappa^{i,i}\right) \middle| \frac{\ell^{n,\cdot}_{T_n}}{\sqrt{n}}\right) \tag{C.3}$$

provided the right hand side is well defined. The subscript "as" means asymptotically. $J(g)$ is then given by

$$J(g) = \frac{1}{2}\mathrm{E}\varrho^{i,j}(Z)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}g(Z) \tag{C.4}$$

We shall also need the following part of Proposition 3 of Mykland(1995):

Proposition C.0.2.1. *Suppose that condition (I3) of Theorem C.0.2.1 holds. Then conditions (H1) and (H2) on* $(\ell^{n,i},\ell^{n,i})_{T_n}$ *are equivalent to the same conditions imposed on either* $[\ell^{n,i},\ell^{n,i}]_{T_n}$ *or* $<\ell^{n,i},\ell^{n,i}>_{T_n}$. *Suppose furthermore that* $<\ell^{n,i},\ell^{n,j},\ell^{n,k}>_{uT_n}/c_n^{3/2}r_n$ *converges in probability to a constant for each* $u\in[0,1]$, *with* $<\ell^{n,i},\ell^{n,j},\ell^{n,k}>_{uT_n}/c_n^{3/2}r_n \to \eta^{i,j,k}$. *Then, if* $\kappa^{i,j}$ *is nonsingular, the following is valid:*

$$\frac{(\ell^{n,i},\ell^{n,j})_{T_n} - <\ell^{n,i},\ell^{n,j}>_{T_n}}{c_n r_n} = \frac{1}{3}\eta^{i,j,k}\kappa_{k,\alpha}c_n^{-1/2}\ell^{n,\alpha}_{T_n} + \frac{1}{3}U_n \tag{C.5}$$

where $U_n = O_{\mathrm{P}}(1)$ *and asymptotically has zero expectation given* $c_n^{-1/2}\ell^n_{T_n}$

Let us consider $\ell^n_t = \sqrt{n}X_t^{*(n)}$ where $X_t^{*(n)}$ is defined in Appendix B. Let $\kappa = \lambda T$, $T_n = T$, $c_n = n, r_n = 1/n$. We shall suppress the i in the notation, because we are in the univariate case. Then

$$[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n} = n^2 \int_0^T \left[\log\left(1 + \frac{Y_u}{N_{u-}}\right)\right]^4 d\xi_u$$

$$\begin{aligned}
\mathrm{E}[\ell^n, \ell^n, \ell^n, \ell^n]_{T_n} &= n^2 \int_0^T \mathrm{E}\left[\left(\frac{1}{N_{u-}}\right)^4 N_u^2 \lambda\right] du + o(1) \\
&= \int_0^T \mathrm{E}\left(\frac{1}{S_{u-}^2}\right) \lambda du + o(1) \\
&= O(c_n^2 r_n^2)
\end{aligned}$$

So condition (I3) is satisfied.

$$\begin{aligned}
< \ell^n, \ell^n, \ell^n >_{uT_n} / c_n^{3/2} r_n &= \frac{n^{3/2}}{\sqrt{n}} \int_0^T \mathrm{E}\left[\left\{\log\left(1 + \frac{Y_u}{N_{u-}}\right)\right\}^3 \middle| \mathcal{F}_{u-}\right] N_{u-}^2 \lambda du \\
&= \frac{1}{n}(\rho - \frac{3\lambda}{2}) \int_0^T \frac{1}{S_{u-}^2} du + o(1/n)
\end{aligned}$$

This implies η of Proposition C.0.2.1 is 0. Hence we can replace $(\ell^n, \ell^n)_{T_n}$ by $< \ell^n, \ell^n >_{T_n}$ in the definition of ϱ.

$< \ell^n, \ell^n >_T$
$= n \int_0^T \mathrm{E}\left[\left\{\log\left(1+\frac{Y_u}{N_{u-}}\right)\right\}^2 \middle| \mathcal{F}_{u-}\right] N_{u-}^2 \lambda du$
$= n \int_0^T \left[p_{u,N_{u-}} \left\{\log\left(1+\frac{1}{N_{u-}}\right)\right\}^2 + (1-p_{u,N_{u-}})\left\{\log\left(1+\frac{1}{N_{u-}}\right)\right\}^2\right] N_{u-}^2 \lambda du$
$= n \int_0^T \left[\frac{1}{N_{u-}^2} + (1-2p_{u,N_{u-}})\frac{1}{N_{u-}^3} + \frac{2}{3N_{u-}^4} + \frac{1}{4N_{u-}^4}\right] N_{u-}^2 \lambda du + \text{smaller order terms}$
$= n\lambda T + (\frac{2}{3} + \frac{1}{4} - \frac{\rho}{\lambda}) n\lambda \int_0^T \frac{1}{N_{u-}^2} du + \text{smaller order terms}$

$$\begin{aligned} \varrho &= \mathrm{E}_{\mathrm{as}}\left(n\left(\frac{< \ell^n, \ell^n >_T}{n} - \lambda T\right) \middle| X_T^{*(n)}\right) \\ &= \left(\frac{11}{12}\lambda - \rho\right) \mathrm{E}_{\mathrm{as}}\left(\frac{1}{n}\int_0^T \frac{1}{S_{u-}^{(n)2}} du \middle| X_T^{*(n)}\right) \\ &= 0 \end{aligned}$$

Hence, if there exist constants $\underline{k}, \bar{k}$ satisfying assumptions (I1) and (I2), then for any $g \in \mathcal{C}$,

$$\mathrm{E}g(X_T^{*(n)}) \quad = \quad \mathrm{E}g(N(0, \lambda T)) + o(1/n)$$

Appendix D

D.1 Proof of Lemma 4.1.0.1

From (6.10) of Mykland(1994)

$$d < S >_t = E(\Delta S_t \mid \mathcal{F}_{t-}) N_t \lambda dt = c\nu N_t \lambda dt = \rho S_t dt$$

Hence, $M_t = S_t - \int_0^t \rho S_u du$ is a martingale. By Ito's formula,

$$d(e^{-\rho t} S_t) = e^{-\rho t}(dS_t - \rho S_t dt)$$

$e^{-\rho t} S_t = \int_0^t e^{-\rho u} dM_u$ is a martingale. □

D.2 Derivation of $\phi(u,t)$ in Equation 3.1

The Kolmogorov's forward equations are:

$$\begin{aligned} \mathrm{P}'_k(t) &= -k\lambda_t \mathrm{P}_k(t) + (k-1)\lambda_t p_t \mathrm{P}_{k-1}(t) + (k+1)\lambda_t(1-p_t)\mathrm{P}_{k+1}(t) \quad k \geq 1 \\ \mathrm{P}'_0(t) &= \lambda_t(1-p_t)\mathrm{P}_1(t) \quad k \geq 1 \end{aligned}$$

Multiplying both sides by u^k and taking sum over k, we get

$$\frac{\partial \phi}{\partial t} - (-\lambda_t u + \lambda_t p_t u^2 + (\lambda_t(1-p_t))\frac{\partial \phi}{\partial u} = 0$$

The corresponding ordinary characteristic differential equation is:

$$\frac{du}{dt} = -(-\lambda_t u + \lambda_t p_t u^2 + (\lambda_t(1-p_t))$$

Substituting $v = u - 1$ and rearranging terms, we get:

$$dv + (2p_t - 1)\lambda_t v dt = -\lambda_t p_t v^2 dt$$

Let $a_t = \int_0^t exp\{(2p_s - 1)\lambda_s\} ds$. We can do separation of variables as:

$$\frac{d(va_t)}{(va_t)^2} + \frac{\lambda_t p_t}{a_t} dt = 0$$

Let $b_t = \int_0^t \lambda_s p_s / a_s ds$. The general solution of the characteristic differential equation in implicit form is, therefore, given by:

$$C_1 = b_t - \frac{1}{va_t}$$

where C_1 is an arbitrary constant. Thus the genral solution of $\phi(u,t)$ has the structure:

$$\phi(u,t) = f\left(b_t + \frac{1}{(1-u)a_t}\right)$$

where f is a continuously differentiable function. f can be determined by making use of the initial condition $\phi(u,0) = u^{N_0}$. Hence $f(x) = 1 - 1/x$. So

$$\phi(u,t) = \left(1 - \frac{1}{\frac{1}{(1-u)a_t} + b_t}\right)^{N_0}$$

where $a_t = \int_0^t \exp\{(2p_s - 1)\lambda_s\} ds$ and $b_t = \int_0^t \lambda_s p_s / a_s ds$.

D.3 Difference Equation Results for section 2.4

$$
\begin{aligned}
f_k &= a + bf_{k-1} \\
&= a + b(a + bf_{k-2}) \\
&= a + ab + b^2(a + bf_{k-3}) \\
&= \dots \\
&= a(1 + b + b^2 + \dots + b^{k-1}) \\
&= a\frac{b^k - 1}{b - 1} \\
g_k &= a + bg_{k-1} + cd^{k-1} \\
&= a + cd^{k-1} + b(a + cd^{k-2} + bg_{k-2}) \\
&= a + cd^{k-1} + b(a + cd^{k-2}) + b^2(a + cd^{k-3} + bg_{k-3}) \\
&= a + cd^{k-1} + b(a + cd^{k-2}) + b^2(a + cd^{k-3}) + \dots + b^{k-1}(a + c) \\
&= a(1 + b + b^2 + \dots + b^{k-1}) + cd^{k-1}(1 + \frac{b}{d} + (\frac{b}{d})^2 + \dots + (\frac{b}{d})^{k-1}) \\
&= a\frac{b^k - 1}{b - 1} + c\frac{b^k - d^k}{b - d}
\end{aligned}
$$

Appendix E

E.1 Probability Bounds on Stock price process

Since the discounted stock price is a martingale and interest rate is positive, the stock price process is a sub-martingale. Also, if we consider the closure of the state space on the infinite line and set the transition function as in the proof of Thm VI.2.2 of Doob(1953), then by Thm II.2.4′ of Doob, there is a standard extension of the process which is separable relative to the closed sets. Now, Thm 3.2 of Doob section VII.11 is applicable and we get:

$$\forall\ \epsilon > 0, \quad \epsilon \mathrm{P}\{ \underset{\{o \leq t \leq T\}}{L.U.B.} S_t(\omega) \geq \epsilon\} \leq \mathrm{E}(S_T) = e^{\rho T} S_0$$

E.2 Nonexplosion of number of jumps

In the notation of Kerstind and Klebaner(1995), on the S_t scale,

$$\begin{aligned} m(z) &= \mathrm{E}(\frac{Y_n}{n}) = \frac{\rho}{n\sigma_2} \\ \lambda(z) &= nz\sigma^2 \\ \int_0^\infty \frac{1}{m(z)\lambda(z)} dz &= \int_0^\infty \frac{1}{\rho z} dz = \infty \end{aligned}$$

By Thm 1 of Kersting and Klebaner(1995)$\sum_{n=0}^{\infty}(\lambda(Z_n))^{-1} = \infty$ a. s. This is the necassary and sufficient condition for nonexplosion, see for example Chung(1967), that is there are only finitely many jumps in finite time intervals.

Appendix F

Glossary of Financial Terms

Arbitrage A trading strategy that takes advantage of two or more securities being mispriced relative to each other

Ask Price The price that a dealer is offering to sell an asset

Bid-ask Spread The amount by which the ask price exceeds the bid price

Bid Price Price that a dealer is prepared to pay for an asset

Calibration A method for implying volatility parameters from the prices of actively traded options

Call Option An option to buy an asset at a certain price on a certain date

Derivative An instrument whose price depends on, or is derived from, the price of another asset

Expiration Date The end of life of a contract

Hedge A trade designed to reduce risk

Hedge Ratio A ratio of the size of a position in a hedging instrument to the size of the position being hedged

Implied Volatility Volatility implied from an option price using a model

In-the-money Option Either (a) a call option where the asset price is more than than the strike price or (b) a put option where the asset price is less than than the strike price

No-arbitrage Assumption The assumption that there are no arbitrage opportunities in market prices

Out-of-the-money Option Either (a) a call option where the asset price is less than than the strike price or (b) a put option where the asset price is more than than the strike price

Payoff The cash realized by the holder of a derivative at the end of its life

Put Option An option to sell an asset at a certain price on a certain date

Risk-free rate The rate of interest that can be earned without assuming any risks

Risk-neutral world A world where investors are assumed to require no extra return on average for bearing risks

Strike Price The price at which the underlying asset may be bought or sold in an option contract

Volatility A measure of uncertainty of the return realized on an asset

Appendix G

List of Symbols

S_t	:	Stock price in dollars at time t
c	:	Size of minimum jump(Tick size)
N_t	:	Stock price at time t in units of c
p_{t,N_t}	:	Probability of positive jump in the linear birth-death model
$P_n(t)$	:	$\mathrm{P}(N_t = n)$
$\phi(u,t)$	:	$\mathrm{E}(u^{N_t})$
Y_t	:	Jump size at time t in units of c
$p_{i,k}$	:	$\mathrm{P}(Y_t = i \mid N_t = k)$
σ_t	:	Volatility in the Black-Scholes model
n	:	$1/c$
τ_i	:	i-th jump time
W_u	:	Standard Weiner process
$f(S_T)$	:	Payoff of an option with maturity T
$\gamma(Q)$	:	Expected discounted payoff of an option under the measure Q
$\mathcal{M}$	:	Set of all measures that converge to geometric Brownian motion
$\lambda_t g(S_t)$	:	Jump intensity at time t

ρ_t	:	Risk-free interest rate
$\xi(t)$	:	Underlying jump process(process of event times)
θ_t	:	Unobserved state process of λ_t taking values 0 and 1
ζ_t	:	Jump process associated with θ_t
$\mathbf{Q}$	:	Transition matrix for the θ_t process
$\{\mathcal{G}_t\}$	:	the complete filtration $\sigma(S_u, \lambda_u, 0 \leq u \leq t)$
P	:	Probability measure on $\{\mathcal{G}_t\}$ for the process (S_t, λ_t)
$\pi_i(t)$	:	$\mathrm{P}(\theta_t = i \| \mathcal{F}_t)$
$\mathcal{F}_t$	:	$\sigma(S_u, 0 \leq u \leq t)$
Θ	:	$(\pi(0), q_{12}, q_{21}, \lambda_1, \lambda_2)$
$F_i(x,t)$	:	$\mathrm{E}_{\hat{\Theta}}(X \| \mathcal{G}_t)$ when the stock price is cx and $\lambda_t^* = \lambda_i$.
$G(x,t)$	:	$\mathrm{E}_{\hat{\Theta}}(X \| \mathcal{F}_t) = \pi_0(t)F_0(x,t) + \pi_1(t)F_1(x,t)$

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