Statistical Issues in Finance

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1. Outline

- Introduction to mathematical finance
  - Fundamental Theorem of asset pricing
  - Completeness
  - Black Scholes model for stock prices
  - Option pricing and hedging

- Statistics in option pricing
  - Valuing exotic options: american, asian, russian options using stopping times of Brownian motion or by simulation
  - An application: a nonparametric regression problem
  - Implied volatility - univariate and multivariate
  - Market incompleteness

- Interesting areas
  - Inference for discretely sampled diffusions
  - Microstructure noise
  - Multiple assets
  - Jump detection
2. The Fundamental Theorem of Asset Pricing

**Portfolio:** A portfolio is a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_K)$ of $K$ real numbers. The entry $\theta_j$ represents the number of shares of asset $A_j$ that are owned; if $\theta_j < 0$ then the portfolio is said to be short $|\theta_j|$ shares of asset $A_j$. The value of the portfolio $\theta$ at time $t = 0$ is $V_0(\theta) = \sum_{j=1}^{K} \theta_j S^j_0$, and the value of the portfolio $\theta$ at time $t = 1$ in market scenario $\omega_i$ is $V_1(\theta, \omega_i) = \sum_{j=1}^{K} \theta_j S^j_1(\omega_i)$ where $S^j_1(\omega_i)$ is the price of asset $A_j$ at time $t$ in scenario $\omega_i$.

**Arbitrage** An arbitrage is a portfolio $\theta$ that makes money from nothing, formally, a portfolio $\theta$ such that

- either $V_0(\theta) \leq 0$ and $V_1(\theta, \omega_j) > 0 \quad \forall j = 1, 2, \ldots, N$

- or $V_0(\theta) < 0$ and $V_1(\theta, \omega_j) \geq 0 \quad \forall j = 1, 2, \ldots, N$. 
Equilibrium Measure: A probability distribution $\pi_i = \pi(\omega_i)$ on the set of possible market scenarios $\Omega$ is said to be an equilibrium measure (or risk-neutral measure) if, for every asset $A$, the share price of $A$ at time $t = 0$ is the discounted expectation, under $\pi$ of the share price at time $t = 1$, that is, if

$$S_0^j = e^{-r} \sum_i \pi(\omega_i)S_1^j \quad \forall j = 1, 2, \ldots, K.$$

Theorem 1. (Fundamental Theorem of Arbitrage Pricing) There exists an equilibrium measure, called the risk-neutral measure if and only if arbitrages do not exist.

Discounted asset prices are martingales!
3. Completeness

Replicating Portfolios: Consider a market in which there are freely traded assets $B$ and $A_1, A_2, \ldots, A_K$. Denote the share prices of assets $A_j$ and $B$ at time $t$ in market scenario $\omega_i$ by $S^j_t(\omega_i)$ and $S^B_t(\omega_i)$. Say that a portfolio $\theta = (\theta_1, \ldots, \theta_K)$ in the assets $A_1, A_2, \ldots, A_K$ is a replicating portfolio for the asset $B$ if

$$S^B_t(\omega_i) = \sum_{j=1}^{K} \theta_j S^j_t(\omega_i) \quad \forall i = 1, 2, \ldots, N.$$

The importance of replicating portfolios is that they enable financial institutions that sell asset $B$ (for example, options) to hedge: For each share of asset $B$ sold, buy $\theta_j$ shares of asset $A_j$ and hold them to time $t = 1$. Then at time $t = 1$, net gain = net loss = 0. The financial institution selling asset $B$ makes its money (usually) by charging the buyer a transaction fee or premium at time $t = 0$. 
Define a derivative security to be a tradeable asset whose value $V_1$ at time $t = 1$ is a function $V_1(\omega_i)$ of the market scenario. In the language of probability theory, the derivative securities are random variables, as a random variable is defined to be a function of the outcome $\omega_i$.

A market is said to be complete if it has a unique equilibrium measure.

Theorem 2. Completeness Theorem Let $M$ be an arbitrage-free market with a riskless asset. If for every derivative security there is a replicating portfolio in the assets $A_1, A_2, \ldots, A_K$, then the market $M$ is complete. Conversely, if the market $M$ is complete, and if the unique equilibrium measure $\pi$ gives positive probability to every market scenario $\omega_i$, then for every derivative security there is a replicating portfolio in the assets $A_1, A_2, \ldots, A_K$. 
The set of all derivative securities is a vector space: two derivative securities may be added to get another derivative security, and a derivative security may be multiplied by a scalar. The Completeness Theorem states, in the language of linear algebra, that a market is complete if and only if the freely traded assets $A_1, A_2, \ldots, A_K$ span the space of derivative securities.

The financial importance is that, in a complete market, any derivative security may be hedged using a replicating portfolio in the assets $A_1, A_2, \ldots, A_K$. In an incomplete market, there are necessarily derivative securities that cannot be hedged.
4. **Black Scholes model**

In its simplest form, the Black Scholes model involves only two underlying assets, a riskless asset Cash Bond and a risky asset Stock. The asset Cash Bond appreciates at the short rate, or riskless rate of return $r_t$, which generally is assumed to be nonrandom, although possibly time-varying. Thus, the price $B_t$ of the Cash Bond at time $t$ is assumed to satisfy the differential equation

$$\frac{dB_t}{dt} = r_t B_t$$

whose unique solution for the value $B_0 = 1$ is

$$B_t = exp \left( \int_0^t r_s ds \right).$$
The share price $S_t$ of the risky asset Stock at time $t$ is assumed to follow a stochastic differential equation of the form

$$dS_t = \mu_t S_t dt + \sigma S_t dW_t$$

(1)

where $\{W_t\}_{t \geq 0}$ is a standard Brownian motion, $\mu_t$ is a nonrandom (but not necessarily constant) function of $t$, and $\sigma > 0$ is a constant called the volatility of the Stock.

Proposition 1. If the drift coefficient function $\mu_t$ is bounded, then the SDE(1) has a unique solution with initial condition $S_0$, and it is given by

$$S_t = S_0 \exp \left( \sigma W_t - \frac{\sigma^2}{2} t + \int_0^t \mu_s ds \right)$$

Moreover, under the risk-neutral measure, it must be the case that $r_t = \mu_t$.

Corollary 1. Under the risk-neutral measure, the log of the discounted stock price at time $t$ is normally distributed with mean $\log S_0 - \frac{\sigma^2}{2} t$ and variance $\sigma^2 t$. 
A European Call Option on the asset Stock with strike $K$ and expiration date $T$ is a contract that allows the owner to purchase one share of Stock at price $K$ at time $T$. Thus, the value of the Call at time $T$ is $(S_T - K)_+$. 

According to the Fundamental Theorem of Arbitrage Pricing, the price of the asset Call at time $t = 0$ must be the discounted expectation, under the risk-neutral measure, of the value at time $t = T$, which, by Proposition 1, is

$$C(S_0, 0) = C(S_0, 0; K, T) = E(S_T^* - K/B_T)_+$$

where $S_T^*$ has the distribution specified in Corollary 1.

A routine calculation, using integration by parts, shows that $C(x, 0; K, T)$ may be rewritten as

$$C(x, 0; K, T) = x\Phi(z) - K/B_T\Phi(z - \sigma\sqrt{T})$$

where $z = \frac{\log(xB_t/K) + \sigma^2 t/2}{\sigma\sqrt{T}}$.
Using Ito’s formula and the Fundamental Theorem of asset pricing, it can be shown that we get is a self financing portfolio for replicating a call option using the following Hedging Strategy: At time $t \leq T$, hold

\[
C_x(S_t, t) \text{ shares of Stock}
\]

and \((-S_tC_x(St, t) + C(S_t, t))/B_t\) shares of Cash Bond

A portfolio in the assets Cash Bond and Stock consists of a pair of adapted processes \(\{\alpha_t\}_{0 \leq t \leq T}\) and \(\{\beta_t\}_{0 \leq t \leq T}\), representing the number of shares of Cash Bond and Stock that are owned (or shorted) at times \(0 \leq t \leq T\).

The portfolio is said to be self financing if, with probability 1, for every \(t \in [0, T]\),

\[
\alpha_t B_t + \beta_t S_t = \alpha_0 B_0 + \beta_0 S_0 + \int_0^t \alpha_s dB_s + \int_0^t \beta_s dS_s
\]

A portfolio \((\alpha_t, \beta_t)_{0 \leq t \leq T}\) replicates a derivative security whose value at \(t = T\) is \(V_T\) if, with probability 1,

\[
V_T = \alpha_T B_T + \beta_T S_T
\]
5. Exotic options

**American option**: An option contract that can be exercised at any time from the date of purchase up to and including the expiration date.

**Barrier option**: An option whose payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price.

**Asian option**: An option whose payoff depends on the average price of the underlying asset over a certain period of time as opposed to at maturity.

**Russian Option**: An option whose payoff depends on the maximum of the stock price over a certain period of time.
Pricing, optimal exercise time and hedging of these options involve hitting times of Brownian motion, Doob Meyer decomposition of supermartingales, Ito calculus etc. All problems are not exactly solvable especially if they involve multiple assets or discontinuous asset prices and entail simulations, numerical integration, neural networks, nonparametric regression etc.

References:

6. An Example

For a Russian option, the payoff is $V_T = g(M_T)$ where $M_T = \max_{0 \leq t \leq T} S_t$. Hence the price at time $t = 0$ is $V_0 = e^{-rT} E g(M_T)$.

It can be shown using Girsanov’s theorem that this equals

$$V_0 = f(S_0, T, g) = e^{-rT} E g \left( \exp(\log S_0 + \sigma \max_{0 \leq t \leq T} X_t) \right) \exp(\nu X_T - \frac{1}{2} \nu^2 T)$$

where $X_t$ is Brownian motion and $\sigma \nu = r - \sigma^2/2$.

Now we want the price at a time $t \in (0, T)$.
That is $V_t = e^{-r(T-t)} E(g(M_T) \mid \mathcal{F}_t)$.
This is a function of $(M_t, S_t, T - t)$ which may be impossible to evaluate analytically.
In general, $V_t = E(V_T \mid \mathcal{F}_t) = E(V_T \mid M_t, S_t) = f(M_t, S_t, T - t)$ where $f$ is an unknown function.

The hedge ratio is $f'_S(M_t, S_t, T - t)$.

- You know that $V_t = E(V_T \mid M_t, S_t) = f(M_t, S_t, T - t)$
- You don’t know the form of $f$
- You know how to simulate $(V_T; M_t; S_t)$, but not $V_t$.

Procedure:
Simulate $n$ Copies $(V_T^{(i)}; M_t^{(i)}; S_t^{(i)}) \quad i = 1, \ldots, n$.
Since $f(m; s; T - t) = E(V_T \mid M_t = m; S_t = s)$,
use estimate $\hat{f}(m; s; T-t) = \text{Nonparametric Regression of } V_T^{(i)} \text{ on } M_t^{(i)}, S_t^{(i)}$.

Hedging involves computing the partial derivative of this nonparametric regression estimate.
7. Implied volatility

Recall that the Black Scholes option price is given by:

\[ C(x, 0; K, T) = x\Phi(z) - K/B_T\Phi(z - \sigma\sqrt{T}) \]

where \( z = \frac{\log(xB_t/K) + \sigma^2 t/2}{\sigma\sqrt{T}} \).

Everything in this expression is known except for \( \sigma \).

The value of \( \sigma \) obtained by equating the RHS of the above expression to the observed market price of the call option is called implied volatility.

The problem is that options with different strikes and expirations give different values of \( \sigma \). Hence in practice there is a volatility surface \( \sigma(K, T) \). It has been observed that this is an illposed inverse problem.
The problem becomes more difficult for options whose values depend on multiple stocks eg Basket options. There we also have the question of correlation or comovement of many assets. Some solutions: Copulas, Weighted Monte Carlo.
8. Market incompleteness

References:

- Follmer, Schweizer Hedging of contingent claims under incomplete information
- Schweitzer (1993) Semimartingales and Hedging in incomplete markets Th Prob Appl
9. Inference for discretely sampled diffusions

Consider a continuous time parametric diffusion

\[ dX_t = \mu(X_t; \theta)dt + \sigma(X_t; \theta)dW_t \]

where \( W_t \) is standard Brownian motion, \( \mu(\cdot; \cdot) \) and \( \sigma(\cdot; \cdot) \) are known functions and \( \theta \) an unknown parameter vector. While the data is written in continuous time, the available data is in discrete time. Ignoring this difference can result in inconsistent estimators (Melino 1994). Choices:

- Methods based on simulations (Gallant and Tauchen 1996)
- Nonparametric density matching (Ait-Sahalia 1996)
- Nonparametric regression of approximate moments (Stanton 1997)
- Bayesian (Jones 1997)
Methods based on likelihood: Likelihood cannot be determined explicitly. Log-likelihood of the transition function is not available in closed form.

  Calculate expressions for the transition function in terms of functionals of the Brownian Bridge.

To compute likelihood function, need to solve PDE numerically or simulate large number of paths over which process is sampled very finely. Neither method produces closed form expression to be maximized over $\theta$.

  Construct closed form sequences of approximations to the transition density.

Provide general method to analyze the asymptotic properties of a variety of estimators for discretely sampled continuous time diffusion with random observation times.
10. Microstructure Noise

In theory, the sum of squares of log returns sampled at high frequency estimates the integrated volatility. However in high frequency data, the transaction price is the efficient price plus some noise component due to imperfections in the trading process (Black 1986). When market microstructure noise is present and unaccounted for, the optimal sampling frequency is finite. This involves throwing away a lot of data.

  Assume parametric structure on volatility and model the noise.

- Zhang, Mykland and Ait-Sahalia. forthcoming JASA.
  If volatility is a stochastic process with no parametric structure, use subsampling and averaging, involving estimators constructed on 2 time scales.
  Kernel-based estimators, modified to get rid of end effects, which is necessary for consistency. The optimal estimator converges to the integrated variance at rate $m^{1/4}$ where $m$ is the number of intraday returns.

• Hansen and Lunde. In Preparation
  Show empirically that microstructure noise is time-dependent and correlated with increments in the efficient price. Apply cointegration techniques to decompose transaction prices and bid-ask quotes into estimate of efficient price and noise.
11. Multiple assets

Multivariate methods become important for portfolio management or computing risk metrics like Value at risk. Risk management is right now very hot in the financial institutions. Here we need to consider interest rates, foreign exchange, indices like S&P500 in addition to stocks.

One approach is to use multivariate normality and estimate covariance. Covariance estimation has all the problems associated with volatility estimation plus problems that arise in multivariate time series like cointegration. Another complication is nonsynchronicity.

Another popular semiparametric approach is to use copulas.

References

12. Jump Detection

The main interest now is in risk management: evaluating quantities like value at risk which measure the variability of a portfolio and are expressed in terms of integrated volatility (called realized volatility).

From empirical studies it is seen that jumps have almost no predictive power for realized volatility measurements.

If we use sum of squares of returns as a measure of volatility, it is highly affected by jumps.

How to disentangle jumps from the continuous component? Does microstructure noise make jump detection more difficult? Does frequency of observation have an effect on jump detection?