# Extrema of Gaussian fields on growing manifolds

Wanli Qiao *
Department of Statistics
University of California
One Shields Ave.
Davis, CA 95616-8705
USA

Wolfgang Polonik
Department of Statistics
University of California
One Shields Ave.
Davis, CA 95616-8705
USA

November 24, 2014

**Abstract**

We consider a class of non-homogeneous, continuous, centered Gaussian random fields $\{X_h(t), t \in \mathcal{M}_h; 0 < h \leq 1\}$ where $\mathcal{M}_h$ denotes a rescaled smooth manifold, i.e. $\mathcal{M}_h = \frac{1}{h}\mathcal{M}$, and study the limit behavior of the extreme values of these Gaussian random fields when $h$ tends to zero. Our main result can be thought of as a generalization of a classical result of Bickel and Rosenblatt (1973a), and also of results by Mikhaleva and Piterbarg (1997). This work can be considered as a companion paper to the theoretical study of a nonparametric filament estimator using kernel density estimation that is conducted in Qiao and Polonik (2014).

This research was partially support by the NSF-grant DMS 1107206
*AMS 2000 subject classifications.* Primary 60G70, 60G15.
*Keywords and phrases.* Local stationarity, extreme values, triangulation of manifolds

*The majority of the work was done while Wanli Qiao was a PhD-student at the University of California, Davis. Now he is Quantiative Financial Analyst at Bank of America.

# 1 Introduction

In a classical paper Bickel and Rosenblatt (1973a) derive the asymptotic distribution of the quantity $\sup_{x\in\mathbb{R}} |\widehat{f}_n(x) - f(x)|$, where $\widehat{f}_n$ is a kernel density estimator based on a sample of independent observations from $f$. The case of a multivariate kernel density estimator was treated later in Rosenblatt (1976). These derivations rely heavily on an approximation of $\widehat{f}_n(x) - f(x)$ by a Gaussian processes. This approximation then motivates the consideration of Gaussian processes of the form $\mu_T(t) + Y_T(t)$ with $\mu_T(t)$ deterministic and $\{Y_T(t), 0 \le t \le T\}$ stationary, and it turned out that the behavior as $T \to \infty$ of the supremum of these Gaussian processes was a crucial ingredient to the proofs of these classical results. A similar idea underlies the derivations in Qiao and Polonik (2014). There, however, the Euclidean space is replaced by a smooth manifold, and this is the set-up in this paper as well. In fact, a special case of our main result serves as a crucial probabilistic ingredient to this work.

Our main result concerns the behavior of the distribution of the suprema of certain non-homogeneous, continuous, centered Gaussian random fields $\{X_h(t), t \in \mathcal{M}_h; 0 < h \le 1\}$ as $h \to 0$, where $\mathcal{M}_h$ denotes a rescaled smooth, compact manifold. This result can be considered as a generalization of the classical Bickel and Rosenblatt result as well as of a result by Mikhaleva and Piterbarg (1997) who considered a *fixed* manifold. Our proof combines ideas from both Bickel and Rosenblatt (1973a) and Mikhaleva and Piterbarg (1997).

The set-up is as follows. Let $r, n \in \mathbb{Z}^+$ with $n \ge 2$ and $1 \le r < n$. Let $\mathcal{H}_1 \subset \mathbb{R}^n$ be a compact set and $\mathcal{M}_1 \subset \mathcal{H}_1$ be a $r$-dimensional Riemannian manifold with “bounded curvature”, the explicit meaning of which will be addressed later. For $0 < h \le 1$ let $\mathcal{H}_h := \{t : ht \in \mathcal{H}_1\}$ and $\mathcal{M}_h := \{t : ht \in \mathcal{M}_1\}$. Further let

$$\{X_h(t), t \in \mathcal{M}_h; 0 < h \le 1\} \tag{1.1}$$

denote a class of non-homogeneous, continuous, centered Gaussian fields indexed by $\mathcal{M}_h, 0 < h \le 1$. Our goal is to derive conditions assuring that for each $z > 0$ we can find $\theta = \theta_h(z)$ with

$$\lim_{h\to 0} \mathbb{P}\Big\{ \sup_{t\in\mathcal{M}_h} |X_h(t)| \le \theta \Big\} = \exp\{-2\exp\{-z\}\}.$$

# 2 Main Result

We first introduce some notation and definitions for Gaussian random fields defined on manifolds. Let $\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$, $\Phi(u) = \int_{-\infty}^{u} \phi(v)dv$, $\bar{\Phi}(u) = 1 - \Phi(u)$ and $\Psi(u) = \phi(u)/u$. Let $\|\cdot\|$ be the $L_2$ norm. For $0 < \alpha \le 2$, let $\chi_\alpha(t)$ be a continuous Gaussian field with $\mathbb{E}\chi_\alpha(t) = -\|t\|^\alpha$ and $\mathrm{Cov}(\chi_\alpha(t), \chi_\alpha(s)) = \|t\|^\alpha + \|s\|^\alpha - \|t - s\|^\alpha$ where $s, t \in \mathbb{R}^n$. The existence of such a field $\chi_\alpha(t)$ follows from Mikhaleva and Piterbarg (1997).

For any compact set $\mathcal{T} \subset \mathbb{R}^n$ define

$$H_\alpha(\mathcal{T}) = \mathbb{E} \exp\Big(\sup_{t\in\mathcal{T}} \chi_\alpha(t)\Big).$$

Next we adopt some notation from Piterbarg and Stamatovich (2001). Let $D$ be a non-degenerated $n \times n$ matrix. For a set $A \subset \mathbb{R}^n$ let $DA = \{Dx, x \in A\}$ denote the image of $A$ under $D$. For any $q > 0$, we let

$$[0,q]^r = \{t : t_i \in [0,q], i = 1, \cdots, r;\ t_i = 0, i = r+1, \cdots, n\},$$

denote a cube of dimension $r$ generated by the first $r$ coordinates in $\mathbb{R}^n$. Let

$$H_\alpha^{D\mathbb{R}^r} := \lim_{q\to\infty} \frac{H_\alpha(D[0,q]^r)}{\lambda_r(D[0,q]^r)},$$

where $\lambda_r$ denotes Lebesgue measure in $\mathbb{R}^r$. It is known that $H_\alpha^{D\mathbb{R}^r}$ exists and $0 < H_\alpha^{D\mathbb{R}^r} < \infty$ (see Belyaev and Piterbarg, 1972). With $D = I$ the unit matrix, we write $H_\alpha^{(r)} = H_\alpha^{I\mathbb{R}^r}$. Since by definition the random field $\chi_\alpha(\cdot)$ is isotropic, $H_\alpha^{D\mathbb{R}^r} = H_\alpha^{(r)}$ for any orthogonal matrix $D$. The constant $H_\alpha := H_\alpha^{(n)}$ is the (generalized) *Pickands constant*.

Further, for positive integers $l$ and $\gamma > 0$, let

$$\begin{aligned} C^r(l,\gamma) &= \{t\gamma : t_i \in [0,l] \cap \mathbb{N}_0, i = 1, \cdots, r; t_i = 0, i = r+1, \cdots, n\} \\ &= \gamma\left([0,l]^r \cap \mathbb{N}_0^n\right), \end{aligned}$$

let $H_\alpha^{D,(r)}(l,\gamma) = H_\alpha(DC^r(l,\gamma))$. Again, for $D$ orthogonal and due to isotropy of $\chi_\alpha(\cdot)$, we just write $H_\alpha^{(r)}(l,\gamma) = H_\alpha^{D,(r)}(l,\gamma)$. We let

$$H_\alpha^{(r)}(\gamma) = \lim_{l\to\infty} \frac{H_\alpha^{(r)}(l,\gamma)}{l^r}$$

assuming this limit exists, and for $r = n$ we simply write $H_\alpha(l,\gamma)$ and $H_\alpha(\gamma)$ instead of $H_\alpha^{(n)}(l,\gamma)$ and $H_\alpha^{(n)}(\gamma)$, respectively.

The following definition can be found in Mikhaleva and Piterbarg (1997), for instance.

**Definition 2.1** (Local $(\alpha, D_t)$-stationarity). *A non-homogeneous random field $X(t), t \in \mathcal{S} \subset \mathbb{R}^n$ has a local $(\alpha, D_t)$-stationary structure, or $X(t)$ is locally $(\alpha, D_t)$-stationary, if the covariance function $r(t_1, t_2)$ of $X(t)$ satisfies the following property. For any $s \in \mathcal{S}$ there exists a non-degenerate matrix $D_s$ such that for any $\epsilon > 0$ there exists a positive $\delta(\epsilon)$ with*

$$1 - (1+\epsilon)\|D_s(t_1 - t_2)\|^\alpha \le r(t_1,t_2) \le 1 - (1-\epsilon)\|D_s(t_1 - t_2)\|^\alpha$$

*for $\|t_1 - s\| < \delta(\epsilon)$ and $\|t_2 - s\| < \delta(\epsilon)$.*

Observe that this definition in particular says that $\text{Var}(X(t)) = 1$ for all $t$. Since here we are considering Gaussian random fields indexed by $h$ and study their behavior as $h \to 0$, we will need local $(\alpha, D_t)$-stationarity to hold in a certain sense uniformly in $h$. The following definition makes this precise.

**Definition 2.2** (Local equi-$(\alpha, D_t)$-stationarity)**.** *Consider a sequence of non-homogeneous random fields $X_h(t), t \in \mathcal{S}_h \subset \mathbb{R}^n$ indexed by $h \in \mathbb{H}$ where $\mathbb{H}$ is an index set. We say $X_h(t)$ has a local equi-$(\alpha, D_t^h)$-stationary structure, or $X_h(t)$ is locally equi-$(\alpha, D_t^h)$-stationary, if the covariance functions $r_h(t_1, t_2)$ of $X_h(t)$ satisfy the following property. For any $s \in \mathcal{S}_h$ there exists a non-degenerate matrix $D_s^h$ such that for any $\epsilon > 0$ there exists a positive $\delta(\epsilon)$ independent of $h$ such that*

$$1 - (1+\epsilon)\|D_s^h(t_1 - t_2)\|^\alpha \le r_h(t_1, t_2) \le 1 - (1-\epsilon)\|D_s^h(t_1 - t_2)\|^\alpha$$

*for $\|t_1 - s\| < \delta(\epsilon)$ and $\|t_2 - s\| < \delta(\epsilon)$.*

An example for such a sequence of random fields is given by the fields introduced in Qiao and Polonik (2014) - see (2.5) below.

We will use some further concepts. First we introduce the condition number of a manifold (see also Genovese et al., 2012). For an $r$-dimensional manifold $\mathcal{M}$ embedded in $\mathbb{R}^n$ let $\Delta(\mathcal{M})$ be the largest $\lambda$ such that each point in $\mathcal{M} \oplus \lambda$ has a unique projection onto $\mathcal{M}$, where $\mathcal{M} \oplus \lambda$ denotes the $\lambda$-enlarged set of $\mathcal{M}$, i.e. the union of all open balls of radius $\lambda$ and midpoint in $\mathcal{M}$. $\Delta(\mathcal{M})$ is called *condition number* of $\mathcal{M}$ in some literature. $\Delta(\mathcal{M})$ has an equivalent definition: at each $u \in \mathcal{M}$ let $T_u\mathcal{M}$ denote the tangent space at $u$ to $\mathcal{M}$ and let $T_u^\perp\mathcal{M}$ be the normal space, which is a $n - r$ dimensional hyperplane. Define the *fiber of size* $a$ at $u$ to be $L_a(u, \mathcal{M}) = T_u^\perp\mathcal{M} \cap (\{u\} \oplus a)$. Then $\Delta(\mathcal{M})$ is the largest $\lambda$ such that $L_a(\lambda, \mathcal{M})$ never intersect for all $u \in \mathcal{M}$. A compact manifold embedded in a Euclidean space has a positive condition number, see de Laat (2011), and references therein. A positive $\Delta(\mathcal{M})$ indicates a "bounded curvature" of $\mathcal{M}$. As indicated in Lemma 3 of Genovese et al. (2012), on a manifold with a positive condition number, small Euclidean distance implies small geodesic distance.

We also need the concept of an *$\epsilon$-net*: Given a set $U$ and a metric $d_U$ on $U$, a set $S \subset U$ is an $\epsilon$-net if for any $u \in U$, we have $\inf_{s \in S} d_U(s, u) \le \epsilon$ and for any $s, t \in S$, we have $d_U(s, t) \ge \epsilon$.

Now we state the main theorem of this section. It is an asymptotic result about the behavior of the extreme value of locally equi-$(\alpha, D_t)$-stationary continuous Gaussian random fields indexed by a parameter $h$ as $h \to 0$. As indicated above, it generalizes Theorem 4.1 in Piterbarg and Stamatovich (2001) and Theorem A1 in Bickel and Rosenblatt (1973a). For an $n \times r$ matrix $G$ we denote by $\|G\|_r^2$ the sum of squares of all minors of order $r$.

**Theorem 2.1.** *Let $\mathcal{H}_1 \subset \mathbb{R}^n$ be a compact set and $\mathcal{H}_h := \{t : ht \in \mathcal{H}_1\}$ for $0 < h \le 1$. Let $\{X_h(t), t \in \mathcal{H}_h, 0 < h \le 1\}$ be class of Gaussian centered locally equi-$(\alpha, D_t^h)$-stationary fields with $D_t^h$ continuous in $h \in (0, 1]$ and $t \in \mathcal{H}_h$. Let $\mathcal{M}_1 \subset \mathcal{H}_1$ be a $r$-dimensional compact*

*Riemannian manifold with* $\Delta(\mathcal{M}_1) > 0$ *and* $\mathcal{M}_h := \{t : ht \in \mathcal{M}_1\}$ *for* $0 < h \leq 1$*. Suppose* $\lim_{h\to 0, ht=t^*} D_t^h = D_{t^*}^0$ *uniformly in* $t^* \in \mathcal{H}_1$*, where all the components of* $D_{t^*}^0$ *are continuous and uniformly bounded in* $t^* \in \mathcal{H}_1$*. Further assume there exist positive constants* $C$ *and* $C'$ *such that*

$$0 < C \leq \inf_{\substack{0<h\leq 1, hs\in\mathcal{H}_1 \\ t\in\mathbb{R}^2\setminus\{0\}}} \frac{\|D_s^h\, t\|^\alpha}{\|t\|^\alpha} \leq \sup_{\substack{0<h\leq 1, hs\in\mathcal{H}_1 \\ t\in\mathbb{R}^2\setminus\{0\}}} \frac{\|D_s^h\, t\|^\alpha}{\|t\|^\alpha} \leq C' < \infty. \tag{2.1}$$

*For any* $\delta > 0$*, define*

$$Q(\delta) := \sup_{0<h\leq 1} \{|r_h(x+y, y)| : x + y \in \mathcal{M}_h, y \in \mathcal{M}_h, \|x\| > \delta\}$$

*where* $r_h$ *is the covariance function of* $X_h(t)$*. Suppose for any* $\delta > 0$*, there exists a positive number* $\eta$ *such that*

$$Q(\delta) < \eta < 1, \tag{2.2}$$

*In addition, assume that there exist a function* $v(\cdot)$ *and a value* $\delta_0 > 0$ *such that for any* $\delta > \delta_0$

$$Q(\delta)\Big|[\log(\delta)]^{2r/\alpha}\Big| \leq v(\delta). \tag{2.3}$$

*where* $v$ *is a monotonically decreasing function with* $v(a^p) = O(v(a)) = o(1)$ *and* $a^{-p} = o(v(a))$ *as* $a \to \infty$ *for any* $p > 0$*. For any fixed* $z$*, define*

$$\theta \equiv \theta(z) = \sqrt{2r\log h^{-1}} + \frac{1}{\sqrt{2r\log h^{-1}}}\Bigg[z + \Big(\frac{r}{\alpha} - \frac{1}{2}\Big)\log\log h^{-1} + \log\Bigg\{\frac{(2r)^{r/\alpha - 1/2}}{\sqrt{2\pi}} H_\alpha^{(r)} \int_{\mathcal{M}_1} \|D_s^0 M_s^1\|_r ds\Bigg\}\Bigg], \tag{2.4}$$

*where* $M_s^1$ *is a* $n \times r$ *matrix with orthonormal columns spanning* $\mathcal{T}_s\mathcal{M}_1$*. Then*

$$\lim_{h\to 0} \mathbb{P}\Big\{\sup_{t\in\mathcal{M}_h} |X_h(t)| \leq \theta\Big\} = \exp\{-2\exp\{-z\}\}.$$

**Remarks.**

1. Note that with (2.1), local equi-$(\alpha, D_t^h)$-stationarity is equivalent to

   $$r_h(t_1, t_2) = 1 - \|D_s^h(t_1 - t_2)\|^\alpha + o(\|t_1 - t_2\|^\alpha) \quad \text{as} \quad \|t_1 - t_2\| \to 0,$$

   uniformly for $t_1, t_2 \in \mathcal{H}_h$ and uniformly in $h$.

2. An example of a function $v(\delta)$ satisfying our assumptions is $v(\delta) = \log(\delta)^{-\beta}$ for $\beta > 0$.

3. Qiao and Polonik (2014) use a special case of the above theorem. In that paper a 1-dimensional growing manifold $\mathcal{M}_h$ embedded in $\mathbb{R}^2$ was considered. The Gaussian random field of interest there is

$$U_h(x) = a_1(hx) \int \big(A_1(hx)\big)^T d^2 K(x - s) dW(s), \tag{2.5}$$

where $W$ is a 2-dimensional Wiener process, $A_1 : \mathbb{R}^2 \mapsto \mathbb{R}^3$ and $a_1 : \mathbb{R}^2 \mapsto \mathbb{R}$ are smooth functions, $K : \mathbb{R}^2 \mapsto \mathbb{R}$ is a smooth kernel density function with the unit ball in $\mathbb{R}^2$ as its support, and $d^2$ is an operator such that $d^2 f(x) = \big(f^{(2,0)}(x),\, f^{(1,1)}(x),\, f^{(0,2)}(x)\big)^T$ for any twice differentiable function $f : \mathbb{R}^2 \mapsto \mathbb{R}$. It is shown in Qiao and Polonik (2014) that the assumptions formulated in that paper insure that the processes $U_h(x)$ satisfy the assumptions of our main theorem in the special case of $r = 1$, $n = 2, \alpha = 2$, and $Q(\delta) = 0$ for $\delta > \delta_0$. Observe that of course the function $v(\delta) = \log(\delta)^{-\beta}$ for $\beta > 0$ works in this case. The fact that this special function $Q(\delta)$ can be used there follows from the assumption that the support of $K$ (and its second oriaoder partial derivatives) is bounded. This implies that the covariances of $U_h(x_1)$ and $U_h(x_2)$ become zero once the distance $\|x_1 - x_2\|$ exceeds a certain threshold.

# 3 Proof of Theorem 2.1

The proof is constructing various approximations to $\sup_{t\in\mathcal{M}_h} |X_h(t)|$ that will facilitate the control of the probability $\mathbb{P}(\sup_{t\in\mathcal{M}_h} |X_h(t)| \le \theta)$. Essentially the process $X_h(t)$ on the manifold is linearized by first approximating the manifold locally via tangent planes, and then defining an approximating process on these tangent planes. This idea underlying the proof is typical for deriving extreme value results for such processes (e.g. see Hüsler et al. (2003)). We begin with some preparations.

We inted to partition the manifold $\mathcal{M}_h$ into certain appropriately chosen subsets. To this end, suppose $V_r(\mathcal{M}_1) = \ell$, then $V_r(\mathcal{M}_h) = \ell/h^r$. For a fixed $\ell^* < \ell$, there exists an $\ell^*$-net on $\mathcal{M}_h$ with respect to geodesic distance with cardinality of $O((h\ell^*)^{-r})$. We then construct a Delaunay triangulation using the $\ell^*$-net and divide $\mathcal{M}_h$ into $m_h = O((h\ell^*)^{-r})$ disjoint pieces $\{J_{k,m_h} : k = 1, 2, \cdots, m_h\}$. The norm of the partition is defined as the largest Hausdorff measure of the individual subsets $J_{k,m_h}$. Our construction is such that the norm of this partition is $O(\ell^{*r})$, uniformly in $h$. It is known that for any $r \in \mathbb{N}$ (and for $\ell^*$ small enough) such an $\ell^*$-net and a Delaunay triangulation exist for compact Riemannian manifolds (see e.g. de Laat 2011). (In the case of $r = 1$, the construction just described simply amounts to choosing all the $O(1/h\ell^*)$ many sets $J_{k,m_h}$ as pieces on the curve of length at most $\ell^*$.) One should point out that while $\ell^*$ has to be chosen sufficiently small, it is a constant. In particular this means that it does not tend to zero in this work, and it also does not vary with $h$.

For sufficiently small $\delta > 0$, let $\mathcal{M}_h^{-\delta} \subset \mathcal{M}_h$ be the $\delta$-enlarged neighborhood of union of the boundaries of all $J_{k,m_h}$ using geodesic distance. The minus sign in superscript indicates that this is a 'small' piece that in the below construction will be 'cut out'. We obtain $J^{\delta}_{k,m_h} = J_{k,m_h} \backslash \mathcal{M}^{-\delta}_h$ and $J^{-\delta}_{k,m_h} = J_{k,m_h} \backslash J^{\delta}_{k,m_h}$for $1 \leq k \leq m_h$. Geometrically we envision $J^{-\delta}_{k,m_h}$ as a small strip along the boundaries of $J_{k,m_h}$ (lying inside $J_{k,m_h}$), and $J^{\delta}_{k,m_h}$ is the set that remains when $J^{-\delta}_{k,m_h}$ is cut out of $J_{k,m_h}$. We have $V_r(J^{-\delta}_{k,m_h}) = O(\delta)$, uniformly in $k$ and $h$. The construction of the Delauney triangulation is such that the boundaries of the projections of all the sets $J_{k,m_h}$, $J^{\delta}_{k,m_h}$ and $J^{-\delta}_{k,m_h}$ onto the local tangent planes are null sets, and thus Jordan measurable. This will be used below.

Let $J$ denote one of the sets $J_{k,m_h}$, $J^{\delta}_{k,m_h}$ and $J^{-\delta}_{k,m_h}$, and let $\{S^h_i(J) \subset J, i = 1, \cdots, N_h(J)\}$ be a cover of this piece constructed using the same Delaunay triangulation technique as above, but of course based on a smaller mesh. As above, by controlling the mesh size, we can control the norm of the partition uniformly over $h$, because of the uniform boundedness of the curvatures of the manifolds $\mathcal{M}_h$.

We choose some point $s^h_i$ on $S^h_i(J)$ and orthogonally project $S^h_i(J)$ onto the tangent space of $\mathcal{M}_h$ at the point $s^h_i$. We denote the mapping by $P_{s^h_i}(\cdot)$ or simply $P_{s_i}(\cdot)$ and we let $\tilde{S}^h_i(J) = P_{s_i}(S^h_i(J))$, which, as indicated above are Jordan measurable by construction.

If $J$ is explicitly indicated in the context, then we often drop $J$ in the notation and simply write $S^h_i$ instead of $S^h_i(J)$. For simplicity and generic discussion, we sometimes also omit the index $i$ of $s^h_i$, $S^h_i$ and $\tilde{S}^h_i$.

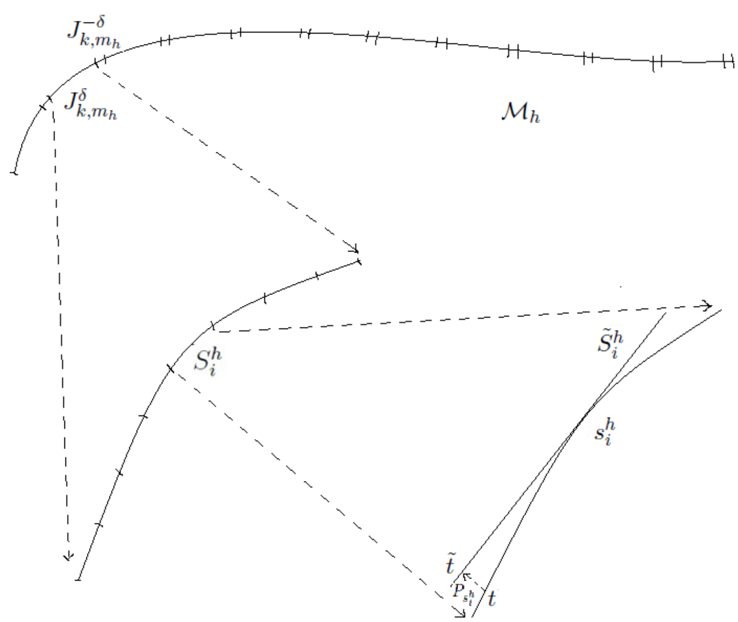


**Figure 3.1:** *This figure visualizes some of the definitions introduced here in the case $r = 1$ and $n = 2$.*

Let $\{M^j_{s^h} : j = 1, \cdots, r\}$ be linearly independent orthonormal vectors spanning the tangent space of $\mathcal{M}_h$ at the point $s^h$, and let $M^h_s$ denote the $n \times r$ matrix with $M^j_{s^h}$ as columns. For a given $\gamma$ consider the (discrete) set $\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h) := \{u : u = s^h + \sum_{j=1}^r i_j \gamma\theta^{-2/\alpha} M^j_{s^h} \in \tilde{S}^h, i_j \in \mathbb{Z}\}$ and let $\Gamma_{\gamma\theta^{-2/\alpha}}(S^h) = (P_{s^h})^{-1}(\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h))$, which is a subset of $S^h$. Note that the geodesic distance between any two adjacent points in $\Gamma_{\gamma\theta^{-2/\alpha}}(S^h)$ is still of the order $O(\gamma\theta^{-2/\alpha})$, again due to the assumed uniformly positive condition number of the manifolds $\mathcal{M}_h$.

The assertion of the theorem is that the probability $\mathbb{P}\{\sup_{t\in\mathcal{M}_h} |X_h(t)| \le \theta\}$ converges to the limit $\exp\{-2\exp\{-z\}\}$ as $h$ tends to zero. This is of course equivalent to say that for any $\epsilon > 0$, we can find a $h_0 > 0$ such that for $0 < h < h_0$, we have

$$\left|\mathbb{P}\Big\{\sup_{t\in\mathcal{M}_h} |X_h(t)| \le \theta\Big\} - \exp\{-2\exp\{-z\}\}\right| < \epsilon.$$

This will be achieved by using various approximations based on the partitions defined above. In the following a high level description of these approximations is provideded. This also outlines the the main ideas of the proof.

**Main ideas of the proof.**

Let $B_h(A) = \{\sup_{t\in A} |X_h(t)| \ge \theta\}$ and as a shorthand notation we use $p_h(A) = \mathbb{P}(B_h(A))$.

**(i)** Approximating $\mathcal{M}_h$ by $\bigcup\limits_{k\le m_h} J^\delta_{k,m_h}$ leads to a corresponding approximation of $p_h(\mathcal{M}_h)$ by $p_h(\bigcup\limits_{k\le m_h} J^\delta_{k,m_h})$. Even though the volume of $\bigcup\limits_{k\le m_h} J^{-\delta}_{k,m_h}$, i.e. the difference of $\mathcal{M}_h$ and $\bigcup\limits_{k\le m_h} J^\delta_{k,m_h}$, tends to infinity as $h \to 0$ if we consider $\delta$ fixed, the difference $p_h(\mathcal{M}_h) - p_h(\bigcup\limits_{k\le m_h} J^\delta_{k,m_h})$ turns out to be of the order $O(\delta)$. Thus we have to choose $\delta$ small enough.

**(ii)** The probabilities $p_h(J^\delta_{k,m_h})$ are approximated by the sum of the probabilities $p_h(S^h_i)$, with $S^h_i$ the cover of $J^\delta_{k,m_h}$ introduced above. The approximation error can be bounded by a double sum, using Bonferroni inequality. To show this double sum is negligible compared with the sum, we have to make sure the volume of $S^h_i$ is not too small, as long as $S^h_i$ is sufficiently small. The double sum will be small if $h$ is small. It turns out that $p_h(J^\delta_{k,m_h})$ essentially behaves like the tail of a normal distribution.

**(iii)** We approximate the small pieces $S^h_i$ on the manifold by $\tilde{S}^h_i$, the projection onto the tangent space and correspondingly approximate the probabilities $p_h(S^h_i)$ by the corresponding probabilities of a transformed field over $\tilde{S}^h_i$. The error generated from the approximation is controlled by choosing the norm of the partitons given by the $S^h_i$ to be sufficiently small.

**(iv)** The probabilities $p_h(\tilde{S}^h_i)$ are approximated by the probabilities $p_h(\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h_i))$, i.e. of the probability of the supremum extended over a collection of dense points on $\tilde{S}^h_i$. The accuracy of the approximation is controlled by choosing both $\gamma$ and $h$ sufficiently small.

**(v)** The sets of dense points considered in (iv) result in a collection $\mathbb{T}_h^\delta$ of dense points in $\bigcup_{k\le m_h} J_{k,m_h}^\delta$. It turns out that the probability $1-p_h(\mathbb{T}_h^\delta) = \mathbb{P}\big(\bigcap_k \big(B_h(\mathbb{T}_h^\delta \cap J_{k,m_h}^\delta)\big)^c\big)$ can be approximated by assuming the events $\big(B_h(\mathbb{T}_h^\delta \cap J_{k,m_h}^\delta)\big)^c$, $k=1,\dots,m_h$, to be independent. To make sure this approximation is valid, $\delta$ may be not too small and $\gamma$ may not too small compared with $h$, as long as both of $\delta$ and $\gamma$ are below the thresholds found in (i) and (iv).

We will now present the detailed proof. We split the proof into different parts in order to provide more structure. Note that the parts do not really follow the logical steps outlined above.

**Part 1**. The various asymptotic approximations in this step are similar to those in the proof of Theorem 1 in Mikhaleva and Piterbarg (1997), but here we consider them in the uniform sense. As indicated above, the uniform boundedness of the curvature of $\mathcal{M}_h$ can be guaranteed due to the boundedness of the curvature (positive condition number) of $\mathcal{M}_1$. For any $\epsilon_1 > 0$, there exists a constant $\delta_1 > 0$ (not depending on $h$) such that if the volumes of all $S_i^h = S_i^h(J_{k,m_h}^\delta)$ are less than $\delta_1$, then we have

$$1-\epsilon_1 \le \frac{V_r(\tilde{S}^h)}{V_r(S^h)} \le 1+\epsilon_1, \tag{3.1}$$

where $V_r(\cdot)$ is the $r$-dimensional Hausdorff measure. On $\tilde{S}^h$ we consider the Gaussian field defined as

$$\tilde{X}_h(\tilde{t}) = X_h(t), \quad \text{with } t \in S^h \text{ such that } \tilde{t} = P_{s_i}(t) \in \tilde{S}^h.$$

Due to local equi-$(\alpha, D_t^h)$-stationarity of $X_h(t)$, for any $\epsilon_2 > 0$, the covariance function $\tilde{r}_h(\tilde{t}_1,\tilde{t}_2)$ of the field $\tilde{X}_h(\tilde{t})$ satisfies

$$1-(1+\epsilon_2/4)\|D_s^h(t_1-t_2)\|^\alpha \le \tilde{r}_h(\tilde{t}_1,\tilde{t}_2) \le 1-(1-\epsilon_2/4)\|D_s^h(t_1-t_2)\|^\alpha$$

for all $t_1, t_2 \in \tilde{S}^h$, if the volume of $S_h$ is less than a certain threshold $\delta_2$, which only depends on $\epsilon_2$. By possibly decreasing $\delta_2$ further we also have

$$1-(1+\epsilon_2/2)\|D_s^h(\tilde{t}_1-\tilde{t}_2)\|^\alpha \le \tilde{r}_h(\tilde{t}_1,\tilde{t}_2) \le 1-(1-\epsilon_2/2)\|D_s^h(\tilde{t}_1-\tilde{t}_2)\|^\alpha$$

for all $\tilde{t}_1, \tilde{t}_2 \in \tilde{S}^h$. Note that this inequality holds uniformly over all $\tilde{S}^h$ under consideration, due to the curvature being bounded on $\mathcal{M}_h$.

On $\tilde{S}^h$ we introduce two homogeneous Gaussian fields $X_h^+(\tilde{t}), X_h^-(\tilde{t})$ such that their covariance functions satisfy

$$r_h^+(\tilde{t}_1,\tilde{t}_2) = 1-(1+\epsilon_2)\|D_s^h(\tilde{t}_1-\tilde{t}_2)\|^\alpha + o(\|D_s^h(\tilde{t}_1-\tilde{t}_2)\|^\alpha)$$

$$r_h^-(\tilde{t}_1,\tilde{t}_2) = 1-(1-\epsilon_2)\|D_s^h(\tilde{t}_1-\tilde{t}_2)\|^\alpha + o(\|D_s^h(\tilde{t}_1-\tilde{t}_2)\|^\alpha)$$

as $\|\tilde{t}_1-\tilde{t}_2\|\to 0$. Thus if the volumes of all $S^h$ under consideration are sufficiently small then

$$r_h^+(\tilde{t}_1,\tilde{t}_2) \le \tilde{r}_h(\tilde{t}_1,\tilde{t}_2) \le r_h^-(\tilde{t}_1,\tilde{t}_2)$$

holds for all $\tilde{t}_1,\tilde{t}_2 \in \tilde{S}^h$. This can be achieved by possibly adjusting $\delta_2$ from above. Slepian's inequality in Lemma 4.1 implies that

$$\begin{aligned}\mathbb{P}\Big(\sup_{\tilde{t}\in\tilde{S}^h} X_h^-(\tilde{t})>\theta\Big) &\le \mathbb{P}\Big(\sup_{\tilde{t}\in\tilde{S}^h} \tilde{X}_h(\tilde{t})>\theta\Big)\\ &= \mathbb{P}\Big(\sup_{\tilde{t}\in S^h} X_h(t)>\theta\Big) \le \mathbb{P}\Big(\sup_{\tilde{t}\in\tilde{S}^h} X^+(\tilde{t})>\theta\Big),\end{aligned}$$

and that

$$\begin{aligned}\mathbb{P}\Big(\max_{\tilde{t}\in\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h)} X_h^-(\tilde{t})>\theta\Big) &\le \mathbb{P}\Big(\max_{\tilde{t}\in\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h)} \tilde{X}_h(\tilde{t})>\theta\Big)\\ &= \mathbb{P}\Big(\max_{t\in\Gamma_{\gamma\theta^{-2/\alpha}}(S^h)} X_h(t)>\theta\Big) \le \mathbb{P}\Big(\max_{\tilde{t}\in\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h)} X_h^+(\tilde{t})>\theta\Big). \qquad (3.2)\end{aligned}$$

For $\tau\in\mathbb{R}^n$ such that $(1+\epsilon_2)^{-1/\alpha}(D_s^h)^{-1}\tau\in\tilde{S}^h$, denote $X_h^+\Big((1+\epsilon_2)^{-1/\alpha}(D_s^h)^{-1}\tau\Big)$ by $Y_h^+(\tau)$ as a function of $\tau$. The covariance function of $Y_h^+(\tau)$ is

$$\begin{aligned}r_{Y_h}^+(\tau_1,\tau_2) &= r_h^+\Big((1+\epsilon_2)^{-1/\alpha}(D_s^h)^{-1}\tau_1,\ (1+\epsilon_2)^{-1/\alpha}(D_s^h)^{-1}\tau_2\Big)\\ &= 1-(1+\epsilon_2)\Big\|D_s^h\Big((1+\epsilon_2)^{-1/\alpha}(D_s^h)^{-1}\tau_1-(1+\epsilon_2)^{-1/\alpha}(D_s^h)^{-1}\tau_2\Big)\Big\|^\alpha + o(\|\tau_1-\tau_2\|^\alpha)\\ &= 1-\|\tau_1-\tau_2\|^\alpha + o(\|\tau_1-\tau_2\|^\alpha)\end{aligned}$$

as $\|\tau_1-\tau_2\|\to 0$. An application of Lemma 4.4 gives that for any $\epsilon_3>0$ and $\theta$ large enough

$$\begin{aligned}&\frac{\mathbb{P}\Big(\max_{\tilde{t}\in\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h)} X_h^+(\tilde{t})>\theta\Big)}{\theta^{2r/\alpha}\Psi(\theta)}\\ &= \frac{\mathbb{P}\Big(\max_{\tau\in(1+\epsilon_2)^{1/\alpha}D_s^h\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h)} X_h^+\Big((1+\epsilon_2)^{-1/\alpha}(D_s^h)^{-1}\tau\Big)>\theta\Big)}{\theta^{2r/\alpha}\Psi(\theta)}\\ &\le \frac{H_\alpha^{(r)}(\gamma)}{\gamma^r}(1+\epsilon_3)V_r((1+\epsilon_2)^{1/\alpha}D_s^h\tilde{S}^h) \qquad (3.3)\\ &= (1+\epsilon_2)^{r/\alpha}(1+\epsilon_3)\frac{H_\alpha^{(r)}(\gamma)}{\gamma^r}\|D_s^hM_s^h\|_rV_r(\tilde{S}^h).\end{aligned}$$

Similarly, by defining $Y_h^-(\tau) = X_h^-\Big((1-\epsilon_2)^{-1/\alpha}(D_s^h)^{-1}\tau\Big)$, we get

$$\frac{\mathbb{P}\Big(\max_{\tilde{t}\in\tilde{\Gamma}_{\gamma\theta^{-2/\alpha}}(\tilde{S}^h)} X_h^-(\tilde{t}) > \theta\Big)}{\theta^{2r/\alpha}\Psi(\theta)} \geq (1-\epsilon_2)^{r/\alpha}(1-\epsilon_3)\frac{H_\alpha^{(r)}(\gamma)}{\gamma^r}\|D_s^h M_s^h\|_r V_r(\tilde{S}^h). \tag{3.4}$$

Combining (3.1), (3.2), (3.3) and (3.4), we obtain for $V_r(S^h)$ small enough and $\theta$ large enough that for any $\epsilon > 0$

$$\begin{aligned}
&\left(1-\tfrac{\epsilon}{4}\right)\frac{H_\alpha^{(r)}(\gamma)}{\gamma^r}\|D_s^h M_s^h\|_r V_r(S^h) \\
&\qquad \leq \frac{\mathbb{P}\Big(\max_{t\in\Gamma_{\gamma\theta^{-2/\alpha}}(S^h)} X_h(t) > \theta\Big)}{\theta^{2r/\alpha}\Psi(\theta)} \\
&\qquad\qquad \leq \left(1-\tfrac{\epsilon}{4}\right)\frac{H_\alpha^{(r)}(\gamma^r)}{\gamma^r}\|D_s^h M_s^h\|_r V_r(S^h),
\end{aligned}$$

and since Lemma 4.6 says that $\frac{H_\alpha^{(r)}(\gamma)/\gamma^r}{H_\alpha^{(r)}} \to 1$ as $\gamma \to 0$, we further have for $\gamma$ sufficiently small that

$$\begin{aligned}
&\left(1-\tfrac{\epsilon}{2}\right)H_\alpha^{(r)}\|D_s^h M_s^h\|_r V_r(S^h) \\
&\qquad \leq \frac{\mathbb{P}\Big(\max_{t\in\Gamma_{\gamma\theta^{-2r/\alpha}}(S^h)} X_h(t) > \theta\Big)}{\theta^{2r/\alpha}\Psi(\theta)} \\
&\qquad\qquad \leq \left(1-\tfrac{\epsilon}{2}\right)H_\alpha^{(r)}\|D_s^h M_s^h\|_r V_r(S^h).
\end{aligned} \tag{3.5}$$

This in fact holds for any $S^h = S_i^h$. We now want to add over $i$. To this end observe that $\sum_{i=1}^{N_h}(\|D_{s_i}^h M_{s_i}^h\|_r V_r(S_i^h))$ is a Riemann sum, namely, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for $\max_{i=1,\cdots,N_h} V_r(S_i^h) < \delta$, we have for $h$ sufficiently small that

$$(1-\epsilon)\int_{J_{k,m_h}^\delta}\|D_s^h M_s^h\|_r ds \leq \sum_{i=1}^{N_h}\|D_{s_i^h}^h M_{s_i^h}^h\|_r V_r(S_i^h) \leq (1+\epsilon)\int_{J_{k,m_h}^\delta}\|D_s^h M_s^h\|_r ds. \tag{3.6}$$

The selection of $\delta$ only depends on $\epsilon$, and the uniformity comes from the fact that as $h \to 0$, $\|D_{t_1}^h - D_{t_2}^h\|_n = \|D_{ht_1}^0 - D_{ht_2}^0\|_n + o(1)$ for any $t_1$ and $t_2$ and that $D_{t^*}^0$ is continuous in $t^* \in \mathcal{H}_1$.

It follows from (3.5) and (3.6) that for any $\epsilon > 0$ and $\gamma, \sup_{0<h\leq 1}\max_{i=1,\cdots,N_h} V_r(S_i^h)$ sufficiently small and $\theta$ large enough

$$(1-\epsilon)H_\alpha^{(r)}\int_{J_{k,m_h}^\delta}\|D_s^h M_s^h\|_r ds \leq \frac{\sum_{i=1}^{N_h}\mathbb{P}\Big(\max_{t\in\Gamma_{\gamma\theta^{-2/\alpha}}(S_i^h)} X_h(t) > \theta\Big)}{\theta^{2r/\alpha}\Psi(\theta)}$$

$$\leq (1+\epsilon)\, H_\alpha^{(r)} \int_{J^\delta_{k,m_h}} \|D_s^h M_s^h\|_r ds. \tag{3.7}$$

Since the distribution of $X_h$ is symmetric, we also have

$$(1-\epsilon)\, H_\alpha^{(r)} \int_{J^\delta_{k,m_h}} \|D_s^h M_s^h\|_r ds \leq \frac{\sum_{i=1}^{N_h} \mathbb{P}\Big( \min_{t\in\Gamma_{\gamma\theta^{-2/\alpha}}(S_i^h)} X_h(t) < -\theta \Big)}{\theta^{2r/\alpha}\Psi(\theta)}$$

$$\leq (1+\epsilon)\, H_\alpha^{(r)} \int_{J^\delta_{k,m_h}} \|D_s^h M_s^h\|_r ds. \tag{3.8}$$

We emphasize that these inequalities hold when the norm of the partition is below a certain threshold that is independent of the choice of $h$.

Following a similar procedure as above we see that the two inequalities above continue to hold (for $h$ and $\max_{i=1,\cdots,N_h} V_r(S_i^h)$ sufficiently small and $\theta$ large enough) if $\max_{t\in\Gamma_{\gamma\theta^{-2/\alpha}}(S_i^h)} X_h(t)$ in (3.7) is replaced by $\sup_{t\in S_i^h} X_h(t)$, and similarly, $\min_{t\in\Gamma_{\gamma\theta^{-2/\alpha}}(S_i^h)} X_h(t)$ in (3.8) is replaced by $\inf_{t\in S_i^h} X_h(t)$. Moreover, if we consider $S_i^h(J_{k,m_h})$ and $S_i^h(J_{k,m_h}\backslash J^\delta_{k,m_h})$, instead of $S_i^h(= S_i^h(J^\delta_{k,m_h}))$, these inequalities continue to hold when we replace $J^\delta_{k,m_h}$ by $J_{k,m_h}$ or $J_{k,m_h}\backslash J^\delta_{k,m_h}$, respectively. In particular for $J_{k,m_h}$ we obtain

$$\frac{\sum_{i=1}^{N_h(J_{k,m_h})} \mathbb{P}\Big( \sup_{t\in S_i^h(J_{k,m_h})} X_h(t) > \theta \Big)}{\theta^{2r/\alpha}\Psi(\theta)} = (1+o(1))\; H_\alpha^{(r)} \int_{J_{k,m_h}} \|D_s^h M_s^h\|_r ds, \tag{3.9}$$

where the $o(1)$-term is uniform in $1 \leq k \leq m_h$ as $\theta \to \infty$.

**Part 2.** Here we consider $J_{k,m_h}$. Let $\{S_i^h : i = 1, \cdots, N_h\}$ denote the partition of $J_{k,m_h}$ constructed at the beginning of the proof via Delauney triangulation. This partition consists of closed non-overlapping subsets, i.e. their interiors are disjoint. Let further

$$B_i = \Big\{ \sup_{t\in S_i^h} X_h(t) > \theta \Big\}.$$

Then obviously,

$$\mathbb{P}\Big( \sup_{t\in J_{k,m_h}} X_h(t) > \theta \Big) = \mathbb{P}\Big( \bigcup_{i=1}^{N_h} B_i \Big).$$

We now use

$$\sum_{i=1}^{N_h} \mathbb{P}(B_i) - \sum_{1\leq i<j\leq N_h} \mathbb{P}(B_i \cap B_j) \leq \mathbb{P}\Big( \bigcup_{i=1}^{N_h} B_i \Big) \leq \sum_{i=1}^{N_h} \mathbb{P}(B_i),$$

and we want to show that the double sum on the left-hand side is negligible as compared to the sum, so that we essentially have upper and lower bounds for $\mathbb{P}\big(\bigcup_{i=1}^{N_h} B_i\big)$ in terms of $\sum_{i=1}^{N_h} \mathbb{P}(B_i)$. To see this, first observe that it follows from (3.9) that for $\max_{i=1,\cdots,N_h} V_r(S_i^h)$ small enough we have as $\theta \to \infty$ that

$$\sum_{i=1}^{N_h} \mathbb{P}(B_i) = O(\theta^{2r/\alpha}\Psi(\theta)). \tag{3.10}$$

We thus want to show that $\sum_{1\le i<j\le N_h} \mathbb{P}(B_i \cap B_j) = o(\theta^{2r/\alpha}\Psi(\theta))$ as $\theta \to \infty$. The proof for a fixed manifold (i.e. $h$ fixed) can be found in the last part of Mikhaleva and Piterbarg (1997). Our proof for the more general case (uniformly in $h$) is following a similar procedure. It will turn out that we obtain the desired result if the norm of the partition given by the $S_i^h$ can be chosen arbitrarily small, uniformly in $h$. It has been discussed at the beginning of the proof that this is in fact the case.

Let $U = \{(i,j) : B_i \text{ and } B_j \text{ are adjacent}\}$ and $V = \{(i,j) : B_i \text{ and } B_j \text{ are not adjacent}\}$, where non-adjacent means that their boundaries do not touch. Note that

$$\sum_{1\le i<j\le N_h} \mathbb{P}(B_i \cap B_j) = \sum_{\substack{1\le i<j\le N_h,\\ (i,j)\in U}} \mathbb{P}(B_i \cap B_j) + \sum_{\substack{1\le i<j\le N_h,\\ (i,j)\in V}} \mathbb{P}(B_i \cap B_j). \tag{3.11}$$

In what follows we discuss the two sums on the right hand side of (3.11). First we consider the case that $S_i^h$, $S_j^h \in U$ are adjacent, i.e. $(i,j) \in U$. The developments in Part 1 are here applied to $S_i^h$, $S_j^h$ and $S_i^h \cup S_j^h$, respectively. We choose the points where the tangent spaces are placed to be the same for $S_i^h$, $S_j^h$ and $S_i^h \cup S_j^h$, i.e., we choose this point to lie on the boundary of both $S_i^h$ and $S_j^h$. Simply denote this point as $s$. Then, by using the results from Part 1, for any $\epsilon > 0$, when $\max_{(i,j)\in U} V_r(S_i^h \cup S_j^h)$ is small enough and $\theta$ is large enough, then the bounds obtained as in Part 1 result in

$$\begin{aligned}\frac{\mathbb{P}(B_i \cap B_j)}{\theta^{2r/\alpha}\Psi(\theta)} &= \frac{\mathbb{P}(B_i) + \mathbb{P}(B_j) - \mathbb{P}(B_i \cup B_j)}{\theta^{2r/\alpha}\Psi(\theta)} \\ &\le (1+\epsilon)H_\alpha^{(r)}\, \|D_s^h M_s^h\|_r\, V_r(S_i^h) + (1+\epsilon)H_\alpha^{(r)}\, \|D_s^h M_s^h\|_r\, V_r(S_j^h) \\ &\qquad - (1-\epsilon)H_\alpha^{(r)}\, \|D_s^h M_s^h\|_r\, V_r(S_i^h \cup S_j^h) \\ &= 2\epsilon H_\alpha^{(r)}\, \|D_s^h M_s^h\|_r\, [V_r(S_i^h) + V_r(S_j^h)].\end{aligned}$$

The sum of the right hand side of the above inequalities over $(i,j) \in U$ again is a Riemann sum that approximates an integral over $J_{k,m_h}$. Since $\lim_{h\to 0, ht=t^*} D_t^h = D_{t^*}^0$ uniformly in $t^* \in \mathcal{H}_1$, and since the components of $D_{t^*}^0$ are continuous and bounded in $t^* \in \mathcal{H}_1$, there exists a finite real $c > 0$ such that

$$\sup_{s\in\mathcal{M}_h, 0<h\le 1} \|D_s^h M_s^h\|_r \le c. \tag{3.12}$$

Hence as $\max_{1\le i\le N_h} V_r(S_i^h) \to 0$ and $\theta \to \infty$, and noting that $\epsilon > 0$ is arbitrary, we have

$$\sum_{\substack{1\le i<j\le N_h,\\ (i,j)\in U}} \mathbb{P}(B_i \cap B_j) = o(\theta^{2r/\alpha}\Psi(\theta)). \tag{3.13}$$

Next we proceed to consider the case that $(i,j) \in V$, i.e. $S_i^h, S_j^h$ are not adjacent on $J_{k,m_h}$. To find a upper bound for $\mathbb{P}(B_i \cap B_j)$, first notice that

$$\begin{aligned} \mathbb{P}(B_i \cap B_j) &= \mathbb{P}\Big(\sup_{t\in S_i^h} X_h(t) > \theta, \sup_{t\in S_j^h} X_h(t) > \theta\Big) \\ &\le \mathbb{P}\Big(\sup_{t\in S_i^h, s\in S_j^h} (X_h(t) + X_h(s)) > 2\theta\Big). \end{aligned} \tag{3.14}$$

In order to further estimate this probability we will use the following Borel theorem from Belyaev and Piterbarg (1972).

**Theorem 3.1.** *Let $\{X(t), t \in T\}$ be a real separable Gaussian process indexed by an arbitrary parameter set $T$, let*

$$\sigma^2 = \sup_{t\in T} \mathrm{Var} X(t) < \infty, \quad m = \sup_{t\in T} \mathbb{E}X(t) < \infty,$$

*and let the real number $b$ be such that*

$$\mathbb{P}\Big(\sup_{t\in T} X(t) - \mathbb{E}X(t) \ge b\Big) \le \frac{1}{2}.$$

*Then for all $x$*

$$\mathbb{P}\Big(\sup_{t\in T} X(t) > x\Big) \le 2\bar{\Phi}\Big(\frac{x-m-b}{\sigma}\Big).$$

There exists a constant $\zeta_1 > 0$ such that

$$\inf_{(i,j)\in V, t\in S_i^h, s\in S_j^h, 0<h\le 1} \|t - s\| > \zeta_1,$$

i.e., the distance between any two nonadjacent elements of the partition exceeds $\zeta_1$ uniformly in $h \in (0,1]$. This is due to the fact that the curvatures of the manifolds $\mathcal{M}_h$ is (uniformly) bounded, and that $V_r(S_j^h)$ is bounded away from zero uniformly in $j$ and $h$. See Lemma 3 of Genovese et al. (2012) for more details underlying this argument. The latter also implies that we can find a number $N_0 > 0$ such that $N_h$, the number of sets $S_i$, satisfies $N_h < N_0$ for all $h$. Assumption (2.2) implies that

$$\rho := \sup_{\|t-s\|\ge\zeta_1, 0<h\le 1} r_h(t,s) < 1.$$

We want to apply the above Borel theorem to $X_h(t) + X_h(s)$ with $t \in S_i^h$ and $s \in S_j^h$ and $(i,j) \in V$. To this end observe that

$$\sup_{0<h\le 1} \sup_{t\in S_i^h, s\in S_j^h} \mathrm{Var}\left(X_h(t) + X_h(s)\right) \le 2 + 2\rho$$

and

$$\sup_{0<h\le 1} \sup_{t\in S_i^h, s\in S_j^h} \mathbb{E}\left(X_h(t) + X_h(s)\right) = 0.$$

Next we show that there is a constant $b$ such that $\mathbb{P}\Big(\sup_{t\in S_i^h, s\in S_j^h}\left(X_h(t) + X_h(s)\right) > b\Big) \le \frac{1}{2}$ for $h$ sufficiently small. Note that

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in S_i^h, s\in S_j^h}\left(X_h(t) + X_h(s)\right) > b\Big) &\le \mathbb{P}\Big(\sup_{t\in J_{k,m_h}, s\in J_{k,m_h}}\left(X_h(t) + X_h(s)\right) > b\Big)\\ &\le \mathbb{P}\Big(\sup_{t\in J_{k,m_h}} X_h(t) > b/2\Big).\end{aligned}$$

All the arguments in **Part1** hold uniformly in $h$ as long as $\theta$ is large enough. In other words, the conclusions there can be restated by replacing $\theta$ with $x$ where $x \to \infty$. For instance, for any $\epsilon > 0$ we can choose $\max_{1\le i\le N_h} V_r(S_i^h)$ small enough such that

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in J_{k,m_h}} X_h(t) > x\Big) &\le \sum_{i=1}^{N_h} \mathbb{P}\Big(\sup_{t\in S_i^h} X_h(t) > x\Big)\\ &\le (1+\epsilon)\, x^{2r/\alpha}\Psi(x) H_\alpha^{(r)} \int_{J_{k,m_h}} \|D_s^h M_s^h\|_r ds\end{aligned}$$

holds for all $1 \le k \le m_h$ and $x > x_0$. Hence, since $x^{2r/\alpha}\Psi(x) \to 0$ as $x \to \infty$, we can find $b$ such that $\mathbb{P}(\sup_{t\in J_{k,m_h}} X_h(t) > b/2) < 1/2$ for all $1 \le k \le m_h$ when $\max_{1\le i\le N_h} V_r(S_i^h)$ is sufficiently small. The above Borel inequality now gives (for large enough $\theta$) that

$$\mathbb{P}\Big(\sup_{t\in S_i^h, s\in S_j^h}\left(X_h(t) + X_h(s)\right) > 2\theta\Big) \le 2\bar{\Phi}\Big(\frac{\theta - b/2}{\sqrt{(1+\rho)/2}}\Big). \tag{3.15}$$

Since the total number of elements in the sum in (3.11) is bounded by $N_h^2$, it follows from (3.14) and (3.15) that uniformly in $k$ (recall that the $B_i$ depend on $k$)

$$\sum_{\substack{1\le i<j\le N_h,\\ j-i>1}} \mathbb{P}(B_i \cap B_j) \le 2N_h^2\bar{\Phi}\Big(\frac{\theta - b/2}{\sqrt{(1+\rho)/2}}\Big) \le 2N_0^2\bar{\Phi}\Big(\frac{\theta - b/2}{\sqrt{(1+\rho)/2}}\Big) = o(\theta^{2r/\alpha}\Psi(\theta)). \tag{3.16}$$

as $\theta \to \infty$ by using the well-known fact that $\lim_{u\to\infty} \frac{\bar{\Phi}(u)}{\Psi(u)} = 1$ (see Cramér, 1951, page 374).

Considering (3.10), (3.11), (3.13) and (3.16) and their respective conditions, we have

$$\mathbb{P}\Big(\sup_{t\in J_{k,m_h}} X_h(t)>\theta\Big) = (1+o(1))\sum_{i=1}^{N_h}\mathbb{P}\Big(\sup_{t\in S_i^h} X_h(t)>\theta\Big) \quad \text{as } \theta\to\infty, \tag{3.17}$$

where the $o(1)$-term is uniform in $k$.

Combining (3.9) and (3.17), we have for $\sup_{h\in(0,1]}\max_{1\le i\le N_h} V_r(S_i^h)$ sufficiently small, that

$$\sum_{k\le m_h}\mathbb{P}\Big(\sup_{t\in J_{k,m_h}} X_h(t)>\theta\Big) = (1+o(1))\theta^{2r/\alpha}\Psi(\theta)H_\alpha^{(r)}\int_{\mathcal{M}_h}\|D_s^hM_s^h\|_r ds \quad \text{as } h\to 0. \tag{3.18}$$

**Part 3.** Note that from the expression of $\theta$ in (2.4) we have for any fixed $z$

$$\theta^{2r/\alpha}\Psi(\theta) = \frac{\theta^{2r/\alpha-1}}{\sqrt{2\pi}}\exp\Big\{-\frac{\theta^2}{2}\Big\} = \frac{h^r\exp\{-z\}}{H_\alpha^{(r)}\int_{\mathcal{M}_1}\|D_s^0M_s^1\|_r ds}(1+o(1)) = O(h^r) \tag{3.19}$$

as $h\to 0$.

Observing that $\max_{1\le k\le m_h} V_r(J_{k,m_h}^{-\delta}) = O(\delta)$ (uniformly in $h$), and using (3.19) we obtain for $h$ small enough that

$$\begin{aligned}
0 &\le \mathbb{P}\Big(\sup_{t\in\mathcal{M}_h} X_h(t)>\theta\Big) - \mathbb{P}\Big(\sup_{t\in\cup_{k\le m_h}J_{k,m_h}^{\delta}} X_h(t)>\theta\Big)\\
&\le \mathbb{P}\Big(\sup_{t\in\mathcal{M}_h\setminus\cup_{k\le m_h}J_{k,m_h}^{\delta}} X_h(t)>\theta\Big)\\
&\le \sum_{k=1}^{m_h}\mathbb{P}\Big(\sup_{t\in J_{k,m_h}^{-\delta}} X_h(t)>\theta\Big)\\
&\le (1+\epsilon)\,\theta^{2r/\alpha}\Psi(\theta)H_\alpha^{(r)}\sum_{k=1}^{m_h}\int_{J_{k,m_h}^{-\delta}}\|D_s^hM_s^h\|_r ds\\
&\le O(\delta)(1+\epsilon)H_\alpha^{(r)}cm_h\theta^{2r/\alpha}\Psi(\theta)\\
&\le O(\delta)(1+\epsilon)\,H_\alpha^{(r)}c\,O((hl^*)^{-r})\frac{h^r\exp\{-z\}}{H_\alpha^{(r)}\int_{\mathcal{M}_1}\|D_s^0M_s^1\|_r ds}\\
&= O(\delta),
\end{aligned}$$

uniformly in $k$. Here $c$ is from (3.12). Similarly, (and again uniformly in $k$) we have $0\le \mathbb{P}\big(\inf_{t\in\mathcal{M}_h}X_h(t)<-\theta\big) - \mathbb{P}\big(\inf_{t\in\cup_{k\le m_h}J_{k,m_h}^{\delta}}X_h(t)<-\theta\big) = O(\delta)$ uniformly in $0<h\le h_1$ for some $h_1>0$. Collecting what we have we get that uniformly in $0<h\le h_1$

$$\mathbb{P}\Big(\sup_{t\in\mathcal{M}_h}|X_h(t)|\le\theta\Big) = \mathbb{P}\Big(\sup_{t\in\cup_{k\le m_h}J_{k,m_h}^{\delta}}|X_h(t)|\le\theta\Big) + O(\delta) \tag{3.20}$$

and

$$\sum_{k=1}^{m_h} \mathbb{P}\Big( \sup_{t \in J_{k,m_h}} |X_h(t)| > \theta \Big) = \sum_{k=1}^{m_h} \mathbb{P}\Big( \sup_{t \in J^\delta_{k,m_h}} |X_h(t)| > \theta \Big) + O(\delta). \tag{3.21}$$

**Part 4.** We index all the $t \in \Gamma_{\gamma\theta^{-2/\alpha}}(S_i^h(J^\delta_{k,m}))$ for all $1 \le k \le m_h$ and $1 \le i \le N_h(J^\delta_{k,m})$, that is, we denote $\bigcup_{1 \le k \le m_h} \Gamma_{\gamma\theta^{-2/\alpha}}(J^\delta_{k,m})$ by $\{t_j, j = 1, \cdots, N_h^*\}$. Our assumptions assure that $N_h^* = O(\frac{\theta^{2r/\alpha}}{h^r \gamma^r})$, because $V_r(\mathcal{M}_h) = O(h^{-r})$ and the 'mesh size' of the curvilinear mesh on $\mathcal{M}_h$ is $O(\frac{\theta^{2/\alpha}}{\gamma^r})$, due to the construction of the triangulation and the uniformly bounded curvature on the manifolds $\mathcal{M}_h$.

With (3.7), (3.12) and (3.19), we have

$$\mathbb{P}\Big( \max_{t_j \in J^\delta_{k,m_h}} |X_h(t_j)| > \theta \Big) = O(h) \tag{3.22}$$

uniformly in $k$ as $h \to 0$. It follows that as $h \to 0$

$$\sum_{k=1}^{m_h} \log\Big(1 - \mathbb{P}\Big( \max_{t_j \in J^\delta_{k,m_h}} |X_h(t_j)| > \theta \Big)\Big) = (1 + o(1)) \sum_{k=1}^{m_h} \mathbb{P}\Big( \max_{t_j \in J^\delta_{k,m_h}} |X_h(t_j)| > \theta \Big). \tag{3.23}$$

It follows from (3.7) and its version with the max over the discrete set replaced by the sup over $t \in S_i^h$ (see discussion given below (3.8)), that for any $\epsilon > 0$ there exists thresholds for $h$, $\gamma$ and the norm of partitions, such that

$$\begin{aligned} 0 &\le \mathbb{P}\Big( \sup_{t \in J^\delta_{k,m_h}} X_h(t) > \theta \Big) - \mathbb{P}\Big( \max_{t_i \in J^\delta_{k,m_h}} X_h(t_i) > \theta \Big) \\ &\le \sum_{j=1}^{N_h} \Big[ \mathbb{P}\Big( \sup_{t \in S_j^h} X_h(t) > \theta \Big) - \mathbb{P}\Big( \max_{t_i \in S_j^h} X_h(t_i) > \theta \Big) \Big] \\ &\le \epsilon\, \theta^{2r/\alpha} \Psi(\theta) H_\alpha^{(r)} \int_{J^\delta_{k,m_h}} \|D_s^h M_s^h\|_r ds, \end{aligned}$$

provided $h$, $\gamma$ and the norm of partitions are smaller then their respective thresholds. Similarly, (3.8) and its corresponding 'continuous' version imply that for $h$ and $\gamma$ smaller than their respective thresholds indicated in **Part 1**, we have

$$0 \le \mathbb{P}\Big( \inf_{t \in J^\delta_{k,m_h}} X_h(t) < -\theta \Big) - \mathbb{P}\Big( \min_{t_i \in J^\delta_{k,m_h}} X_h(t_i) < -\theta \Big) \le \epsilon\, \theta^{2r/\alpha} \Psi(\theta) H_\alpha^{(r)} \int_{J^\delta_{k,m_h}} \|D_s^h M_s^h\|_r ds.$$

Consequently, if $h$ and $\gamma$ and $\max_{1 \le k \le m_h} V_r(J^\delta_{k,m_h})$ are small enough, we have

$$0 \le \mathbb{P}\Big( \sup_{t \in \bigcup_{k \le m_h} J^\delta_{k,m_h}} |X_h(t)| > \theta \Big) - \mathbb{P}\Big( \max_{t_i \in \bigcup_{k \le m_h} J^\delta_{k,m_h}} |X_h(t_i)| > \theta \Big)$$

$$\leq \sum_{k=1}^{m_h} \Big[\mathbb{P}\Big(\sup_{t\in J^{\delta}_{k,m_h}} |X_h(t)| > \theta\Big) - \mathbb{P}\Big(\max_{t_i\in J^{\delta}_{k,m_h}} |X_h(t_i)| > \theta\Big)\Big]$$

$$\leq \sum_{k=1}^{m_h} \Big[\mathbb{P}\Big(\sup_{t\in J^{\delta}_{k,m_h}} X_h(t) > \theta\Big) + \mathbb{P}\Big(\inf_{t\in J^{\delta}_{k,m_h}} X_h(t) < -\theta\Big) - \mathbb{P}\Big(\max_{t_i\in J^{\delta}_{k,m_h}} X_h(t_i) > \theta\Big)$$

$$- \mathbb{P}\Big(\min_{t_i\in J^{\delta}_{k,m_h}} X_h(t_i) < -\theta\Big)\Big]$$

$$\leq 2\epsilon\, \theta^{2r/\alpha}\Psi(\theta)H_{\alpha}^{(r)} \int_{\bigcup_{k\leq m_h} J^{\delta}_{k,m_h}} \|D_s^h M_s^h\|_r ds$$

$$\leq 2\epsilon\, \theta^{2r/\alpha}\Psi(\theta)H_{\alpha}^{(r)} \int_{\mathcal{M}_h} \|D_s^h M_s^h\|_r ds. \tag{3.24}$$

To see the order of the upper bound in (3.24), by the dominated convergence theorem (and using our assumption on the behavior of $D_s^h$) we have

$$\frac{h^r \int_{\mathcal{M}_h} \|D_s^h M_s^h\|_r ds}{\int_{\mathcal{M}_1} \|D_s^0 M_s^1\|_r ds} = \frac{\int_{\mathcal{M}_1} \|D_{s/h}^h M_s^1\|_r ds}{\int_{\mathcal{M}_1} \|D_s^0 M_s^1\|_r ds} \to 1, \quad \text{as} \quad h \to 0. \tag{3.25}$$

As a result of (3.19) and (3.25), we can write for $\max_{1\leq k\leq m_h} V_r(J^{\delta}_{k,m_h})$ small enough that

$$\mathbb{P}\Big(\sup_{t\in\bigcup_{k\leq m_h} J^{\delta}_{k,m_h}} |X_h(t)| \leq \theta\Big) = \mathbb{P}\Big(\max_{t_i\in\bigcup_{k\leq m_h} J^{\delta}_{k,m_h}} |X_h(t_i)| \leq \theta\Big) + o(1) \tag{3.26}$$

and

$$\sum_{k=1}^{m_h} \mathbb{P}\Big(\sup_{t\in J^{\delta}_{k,m_h}} |X_h(t)| > \theta\Big) = \sum_{k=1}^{m_h} \mathbb{P}\Big(\max_{t_i\in J^{\delta}_{k,m_h}} |X_h(t_i)| > \theta\Big) + o(1), \tag{3.27}$$

as $\gamma, h \to 0$.

**Part 5.** This step uses similar ideas as in the proof of Lemma 5.1 in Berman (1971). To find an upper bound for the difference

$$\Big|\mathbb{P}\Big(\max_{t_j\in\bigcup_{k\leq m_h} J^{\delta}_{k,m_h}} |X_h(t_j)| \leq \theta\Big) - \prod_{k\leq m_h} \mathbb{P}\Big(\max_{t_j\in J^{\delta}_{k,m_h}} |X_h(t_j)| \leq \theta\Big)\Big|, \tag{3.28}$$

we are going to apply Lemma 4.5. Define a probability measure $\tilde{\mathbb{P}}$ such that for any $x_{t_j} \in \mathbb{R}$ with $t_j \in \bigcup_{k\leq m_h} J^{\delta}_{k,m_h}$,

$$\tilde{\mathbb{P}}\Big(X_h(t_j) \leq x_{t_j}, t_j \in \bigcup_{k\leq m_h} J^{\delta}_{k,m_h}\Big) = \prod_{k\leq m_h} \mathbb{P}\Big(X_h(t_j) \leq x_{t_j}, t_j \in J^{\delta}_{k,m_h}\Big),$$

i.e., under $\tilde{\mathbb{P}}$ the vectors $(X_h(t_i) : t_i \in J^\delta_{k,m})$ and $(X_h(t_j) : t_j \in J^\delta_{k',m})$ independent for $k \neq k'$. By Lemma 4.5, the difference in (3.28) can be bounded by

$$
\begin{aligned}
&8 \sum_{\substack{k\le m_h, k'\le m_h,\\ k\neq k'}} \sum_{t_i\in J^\delta_{k,m_h}} \sum_{t_j\in J^\delta_{k',m_h}} \int_0^{|r_h(t_i,t_j)|} \phi(\theta,\theta,\lambda)d\lambda \\
&= 8 \sum_{\substack{k\le m_h, k'\le m_h,\\ k\neq k'}} \sum_{t_i\in J^\delta_{k,m_h}} \sum_{t_j\in J^\delta_{k',m_h}} \int_0^{|r_h(t_i,t_j)|} \frac{1}{2\pi(1-\lambda^2)^{1/2}} \exp\Big(-\frac{\theta^2}{1+\lambda}\Big)d\lambda \\
&\le 8 \sum_{\substack{k\le m_h, k'\le m_h,\\ k\neq k'}} \sum_{t_i\in J^\delta_{k,m_h}} \sum_{t_j\in J^\delta_{k',m_h}} \frac{|r_h(t_i,t_j)|}{2\pi(1-(r_h(t_i,t_j))^2)^{1/2}} \exp\Big(-\frac{\theta^2}{1+|r_h(t_i,t_j)|}\Big). \qquad (3.29)
\end{aligned}
$$

For $t_i \in J^\delta_{k,m_h}$ and $t_j \in J^\delta_{k',m_h}$ with $k \neq k'$, it follows from the uniform boundedness of the curvature of the growing manifold that there exists a positive real $\varsigma$ such that $\|t_i - t_j\| \geq \varsigma$, uniformly for all $0 < h \leq 1$. (Similar arguments have been used above already.) Thus we obtain from assumption (2.2) that there exists $\eta > 0$ dependent on $\varsigma$ such that

$$|r_h(t_i,t_j)| < \eta < 1 \qquad (3.30)$$

uniformly in $t_i \in J^\delta_{k,m_h}$ and $t_j \in J^\delta_{k',m_h}$ with $k \neq k'$ and $0 < h \leq 1$.

Let $\omega$ be an arbitrary number satisfying

$$0 < \omega < \frac{2}{(1+\eta)} - 1.$$

We take $\gamma = v(h^{-1})^{1/3r}$ in what follows and divide the triple sum in (3.29) into two parts: In one part the indices $i, j$ are constrained such that $\|t_i - t_j\| < (N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha}$ and for the other part the indices take the remaining values. In the first part, the number of summands in the triple sum is of the order $O((N_h^*)^{\omega+1})$, because there is a total of $O(N_h^*)$ points and for each of this points we have to consider at most $O((N_h^*)^\omega)$ pairs. Taking (3.30) into account, we get the order of the sum in the first part of (3.29)

$$
\begin{aligned}
O\Big((N_h^*)^{\omega+1}\exp\Big\{-\frac{\theta^2}{1+\eta}\Big\}\Big) &= O\Big(\Big(\frac{\theta^{2r/\alpha}}{h^r\gamma^r}\Big)^{1+\omega}\exp\Big\{-\frac{\theta^2}{1+\eta}\Big\}\Big) \\
&= O\Big(\Big(\frac{(\log h^{-1})^{r/\alpha}}{h^r\gamma^r}\Big)^{1+\omega}\exp\Big\{-\frac{2r\log h^{-1}}{1+\eta}\Big\}\Big) \\
&= O\Big(h^{\frac{2r}{1+\eta}-r(1+\omega)}\Big(\log h^{-1}\Big)^{\frac{(1+\omega)r}{\alpha}}\Big(v(h^{-1})\Big)^{-\frac{(1+\omega)\gamma}{3r}}\Big),
\end{aligned}
$$

which tends to zero as $h$ approaches zero.

Then we consider the second part of (3.29) with $\|t_i - t_j\| \geq (N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha}$. Noticing $(1 + |r^h(t_i,t_j)|)^{-1} \geq 1 - |r^h(t_i,t_j)|$ and (3.30), we can have the following bound for the second part of (3.29):

$$8\exp(-\theta^2) \sum_{\substack{k\leq m_h, k'\leq m_h,\\ k\neq k'}} \sum_{\substack{t_i\in J^\delta_{k,m_h}, t_j\in J^\delta_{k',m_h},\\ \|t_i-t_j\|\geq (N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha}}} \frac{|r^h(t_i,t_j)|}{2\pi(1-\eta^2)^{1/2}} \exp\left(\theta^2|r^h(t_i,t_j)|\right). \tag{3.31}$$

By (2.3) and the fact that $\theta^2 = O(\log h^{-1})$, we have that $\sup_{\|t_i-t_j\|\geq (N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha}} \theta^2|r^h(t_i,t_j)| \to 0$ as $h \to 0$. Hence (3.31) is of the order of

$$h^{2r} \sum_{\substack{k\leq m_h, k'\leq m_h,\\ k\neq k'}} \sum_{\substack{t_i\in J^\delta_{k,m_h}, t_j\in J^\delta_{k',m_h},\\ \|t_i-t_j\|\geq (N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha}}} |r^h(t_i,t_j)| \tag{3.32}$$

When $h$ is sufficiently small we have

$$\sup_{\|t_i-t_j\|\geq (N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha}} |r^h(t_i,t_j)| \leq \frac{v((N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha})}{[\log((N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha})]^{2r/\alpha}}. \tag{3.33}$$

Therefore, due to (2.3), (3.32) is of the order

$$\begin{aligned} & O\left(h^{2r}(N_h^*)^2 \frac{v((N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha})}{[\log((N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha})]^{2/\alpha}}\right) \\ & = O\left(\frac{[\log(h^{-1})]^{2r/\alpha} v((N_h^*)^{\omega/r}\gamma\theta^{-2/\alpha})}{\left[\log\left(h^{-\omega}\left([\log(h^{-1})]^{1/\alpha} v(h^{-1})^{-1/3r}\right)^{\omega-1}\right)\right]^{2r/\alpha} v(h^{-1})^{2/3}}\right) \\ & = o(1) \quad \text{as} \quad h \to 0. \end{aligned}$$

Now we have proved (3.29) tends to zero as $h$ goes to zero. So we have with this choice of $\gamma$ that as $h \to 0$

$$\mathbb{P}\Big(\max_{t_j\in\bigcup_{k\leq m_h} J^\delta_{k,m_h}} |X_h(t_j)| \leq \theta\Big) = \prod_{k\leq m_h} \mathbb{P}\Big(\max_{t_j\in J^\delta_{k,m_h}} |X_h(t_j)| \leq \theta\Big) + o(1), \tag{3.34}$$

where $\delta > 0$ is fixed small enough.

Now we collect all the approximations above, including (3.20), (3.21), (3.26), (3.27), (3.34), (3.23), (3.18), (3.19) and (3.25). We have for $\delta > 0$ and $\sup_{h\in(0,1]} \max_{1\leq i\leq N_h} V_r(S_i^h)$ fixed and chosen small enough, and $\gamma = v(h^{-1})^{1/3r}$ that as $h \to 0$

$$\mathbb{P}\Big(\sup_{t\in\mathcal{M}_h} |X_h(t)| \leq \theta\Big)$$

$$
\begin{aligned}
&\overset{(3.20)}{=\!=} \mathbb{P}\Big( \sup_{t\in\bigcup_{k\le m_h} J^{\delta}_{k,m_h}} |X_h(t)| \le \theta\Big) + o(1)\\
&\overset{(3.26)}{=\!=} \mathbb{P}\Big( \max_{t_j\in\bigcup_{k\le m_h} J^{\delta}_{k,m_h}} |X_h(t_j)| \le \theta\Big) + o(1)\\
&=\!= \mathbb{P}\Big( \bigcap_{k\le m_h} \Big( \max_{t_j\in J^{\delta}_{k,m_h}} |X_h(t_j)| \le \theta\Big)\Big) + o(1)\\
&\overset{(3.34)}{=\!=} \prod_{k\le m_h} \mathbb{P}\Big( \max_{t_j\in J^{\delta}_{k,m_h}} |X_h(t_j)| \le \theta\Big) + o(1)\\
&=\!= \exp\Big\{ \sum_{k\le m_h} \log\Big(1-\mathbb{P}\Big( \max_{t_j\in J^{\delta}_{k,m_h}} |X_h(t_j)| > \theta\Big)\Big)\Big\} + o(1)\\
&\overset{(3.23)}{=\!=} \exp\Big\{ -(1+o(1)) \sum_{k\le m_h} \mathbb{P}\Big( \max_{t_j\in J^{\delta}_{k,m_h}} |X_h(t_j)| > \theta\Big)\Big\} + o(1)\\
&\overset{(3.27)}{=\!=} \exp\Big\{ -(1+o(1)) \Big[ \sum_{k\le m_h} \mathbb{P}\Big( \sup_{t\in J^{\delta}_{k,m_h}} |X_h(t)| > \theta\Big) - o(1)\Big]\Big\} + o(1)\\
&\overset{(3.21)}{=\!=} \exp\Big\{ -2(1+o(1)) \sum_{k\le m_h} \mathbb{P}\Big( \sup_{t\in J_{k,m_h}} X_h(t) > \theta\Big)\Big\} + o(1)\\
&\overset{(3.18)}{=\!=} \exp\Big\{ -2(1+o(1))\, \theta^{2r/\alpha}\Psi(\theta)H^{(r)}_{\alpha} \int_{\mathcal{M}_h} \|D^h_s M^h_s\|_r ds\Big\} + o(1).
\end{aligned}
$$

This completes our proof by using (3.19), (3.25).

# 4 Miscellaneous

In this section we collect some miscellaneous results that are needed in the above proof. We present them in a separate section in order to not interrupt the flow of the above proof.

**Lemma 4.1.** ***(Slepian's lemma; see Slepian, 1962)*** *Let $\{X_t,\, t\in T\}$ and $\{Y_t,\, t\in T\}$ be Gaussian processes satisfying the assumptions of Theorem 3.1 with the same mean functions. If the covariance functions $r_X(s,t)$ and $r_Y(s,t)$ meet the relations*

$$
r_X(t,t) \equiv r_Y(t,t), \quad t\in T \qquad\qquad r_X(s,t)\le r_Y(s,t), \quad t,s\in T,
$$

*then for any $x$*

$$
\mathbb{P}\Big\{ \sup_{t\in T} X_t < x\Big\} \le \mathbb{P}\Big\{ \sup_{t\in T} Y_t < x\Big\}.
$$

We also need this result from Piterbarg (1996).

**Lemma 4.2.** ***(Lemma 6.1 of Piterbarg, 1996)*** *Let $X(t)$ be a continuous homogeneous Gaussian field where $t \in \mathbb{R}^n$ with expected value $\mathbb{E}X(t) = 0$ and covariance function $r(t)$ satisfying*

$$r(t) = \mathbb{E}(X(t+s)X(s)) = 1 - \|t\|^\alpha + o(\|t\|^\alpha).$$

*Then for any compact set $\mathcal{T} \subset \mathbb{R}^n$*

$$\mathbb{P}\Big(\sup_{t \in u^{-2/\alpha}\mathcal{T}} X(t) > u\Big) = \Psi(u)H_\alpha(\mathcal{T})(1+o(1)) \quad as \ u \to \infty.$$

The next result follows immediately.

**Corollary 4.1.** *Let $X(t)$ be as in Lemma 4.2. Let $M_k \in \mathbb{R}^n, k = 1, \cdots, n$ be a basis of $\mathbb{R}^n$, $l \in \mathbb{Z}^+$ and $\gamma > 0$. We have with $C^r(l,1)$ as defined on page 2 that*

$$\lim_{x \to \infty} \frac{\mathbb{P}(\max_{(i_1,\ldots,i_n) \in C^r(l,1)} X(\sum_{k=1}^n i_k \gamma x^{-2/\alpha} M_k) > x)}{\Psi(x)} = H_\alpha^{(r)}(l,\gamma).$$

$$\lim_{x \to \infty} \frac{\mathbb{P}(\min_{(i_1,\ldots,i_n) \in C^r(l,1)} X(\sum_{k=1}^n i_k \gamma x^{-2/\alpha} M) < -x)}{\Psi(x)} = H_\alpha^{(r)}(l,\gamma).$$

**Remark.** This is also a simple extension of Lemma A1 of Bickel and Rosenblatt (1973a).

**Lemma 4.3.** ***(Lemma 2.3 of Pickands, 1969)*** *Let $X$ and $Y$ be jointly normal, mean zero with variances 1 and covariance $r$. Then*

$$\mathbb{P}(X > x, Y > x) \le (1+r)\Psi(x)\bigg(1 - \Phi\bigg(x\sqrt{\frac{1-r}{1+r}}\,\bigg)\bigg).$$

The next lemma is an extension of Lemma A3 in Bickel and Rosenblatt (1973a), Lemma 3 and and Lemma 5 of Bickel and Rosenblatt (1973b) and Lemma 2.5 in Pickands (1969). Its proof is also adapted from the three sources.

**Lemma 4.4.** *Let $X(t)$ be a centered homogeneous Gaussian field on $\mathbb{R}^n$ with covariance function*

$$r(t) = \mathbb{E}(X(t+s)X(s)) = 1 - \|t\|^\alpha + o(\|t\|^\alpha).$$

*Let $\mathcal{T}$ be a Jordan measurable set imbedded in a $r$-dimensional linear space with $V_r(\mathcal{T}) = \lambda < \infty$. For $\gamma, x > 0$ let $\mathcal{G}(\mathcal{T}, \gamma, x)$ be a collection of points defining a mesh contained in $\mathcal{T}$ with mesh size $\gamma x^{-2/\alpha}$. Assume*

$$\xi(\|t\|) := \inf_{0 < \|s\| \le \|t\|} \|s\|^{-\alpha}(1 - r(s))/2 > 0 \quad for\ \|t\|\ small\ enough. \tag{4.1}$$

*Then*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\max\{X(t) : t \in \mathcal{G}(\mathcal{T}, \gamma, x)\} > x)}{x^{2r/\alpha}\Psi(x)} = \lambda \frac{H_\alpha^{(r)}(\gamma)}{\gamma^r} \tag{4.2}$$

*and*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\sup\{X(t) : t \in \mathcal{T}\} > x)}{x^{2r/\alpha}\Psi(x)} = \lambda H_\alpha^{(r)} \tag{4.3}$$

*uniformly in $\mathcal{T} \in \mathcal{E}_c$ where $\mathcal{E}_c$ is the collection of all $r$-dimensional Jordan measurable sets with $r$-dimensional Hausdorff measure bounded by $c < \infty$. Similarly,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\inf\{X(t) : t \in \mathcal{T}\} < -x)}{x^{2r/\alpha}\Psi(x)} = \lambda H_\alpha^{(r)}. \tag{4.4}$$

*uniformly in $\mathcal{T} \in \mathcal{E}_c$.*

*Proof.* The results in Lemma 3 and and Lemma 5 of Bickel and Rosenblatt (1973b) are similar but they are only given for two-dimensional squares. It is straightforward to generalized them to hyperrectangles and further to Jordan measurable sets. □

**Theorem 4.1.** ***(Theorem 2 of Piterbarg and Stamatovich, 2001)*** *Let $\{X(t), t \in \mathbb{R}^n\}$ be a Gaussian centered locally $(\alpha, D_t)$-stationary field with a continuous matrix function $D_t$. Let $\mathcal{M} \subset \mathbb{R}^n$ be a smooth compact of dimension $r$. Then*

$$\frac{\mathbb{P}(\sup_{t\in\mathcal{M}} X(t) > x)}{x^{2r/\alpha}\Psi(x)} \to H_\alpha^{(r)} \int_{\mathcal{M}} \|D_s M_s\|_r ds$$

*as $x \to \infty$, where $M_s$ is an $n \times r$ matrix with columns the orthonormal basis of the linear subspace tangent to $\mathcal{M}$ at $s$.*

**Lemma 4.5.** ***(Lemma A4 of Bickel and Rosenblatt, 1973a)*** *Let*

$$\phi(x, y, \rho) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{ -\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right\}.$$

*Let $\Sigma_1 = \{r_{ij}\}, \Sigma_2 = \{s_{ij}\}$ be $N \times N$ nonnegative semi-definite matrices with $r_{ii} = s_{ii} = 1$ for all $i$. Let $X = (X_1, \cdots, X_N)$ be a mean 0 Gaussian vector with covariance matrix $\Sigma_1$ under probability measure $\mathbb{P}_{\Sigma_1}$ or $\Sigma_2$ under $\mathbb{P}_{\Sigma_2}$. Let $u_1, \cdots, u_N$ be nonnegative numbers and $u = \min_j u_j$. Then*

$$|\mathbb{P}_{\Sigma_1}[X_j \le u_j, 1 \le j \le N] - \mathbb{P}_{\Sigma_2}[X_j \le u_j, 1 \le j \le N]| \le 4 \sum_{i,j} \left| \int_{s_{ij}}^{r_{ij}} \phi(u, u, \lambda) d\lambda \right|.$$

Recall the definition of $H_\alpha^{(r)}(\gamma) = \lim_{l\to\infty} \frac{H_\alpha^{(r)}(l,\gamma)}{l^r}$ given at the beginning of section 2. We have the following lemma from Bickel and Rosenblatt (1973b).

**Lemma 4.6.** $H_\alpha^{(r)} = \lim_{\gamma\to 0} \frac{H_\alpha^{(r)}(\gamma)}{\gamma^r}$.

**References**


Belyaev Yu.K. and Piterbarg, V.I. (1972): The asymptotic behavior of the average number of the A-points of upcrossings of a Gaussian field beyond a high level. *Akad. Nauk SSSR*, **203**, 9-12.

Berman, S. (1971): Asymptotic independence of the numbers of high and low level crossings of stationary Gaussian processes. *The Annals of Mathematical Statistics*, **42**(3), 927-945.

Bickel, P. and Rosenblatt, M. (1973a): On Some Global Measures of the Deviations of Density Function Estimates. *The Annals of Statistics*, **1**(6) 1071-1095.

Bickel, P. and Rosenblatt, M. (1973b): Two-Dimensional Random Fields, in *Multivariate Analysis III, P.K. Krishnaiah, Ed.* pp. 3-15, Academic Press, New York, 1973.

Cramér, H. (1951): Mathematical methods of statistics. Princeton Univ. Press, Princeton, N.J..

de Laat, D. (2011): Approximating Manifolds by Meshes: Asymptotic Bounds in Higher Codimension. Master Thesis, University of Groningen.

Genovese, C.R., Perone-Pacifico, M., Verdinelli, I. and Wasserman, L. (2012): Minimax Manifold Estimation. *Journal of Machine Learning Research*, **13**(1) 1263-1291.

Hüsler J., Piterbarg, V. and Seleznjev, O. (2003): On Convergence of the Uniform Norms for Gaussian Processes and Linear Approximation Problems. *The Annals of Applied Probability*, **13**(4) 1615-1653

Leibon G. and Letscher D. (2000): Delaunay Triangulation and Voronoi Diagrams for Riemannian Manifolds. *Proceedings of the Sixteenth Annual Symposium on Computational Geometry, SCG '00*, ACM New York, NY, USA, pp. 341-349.

Mikhaleva, T.L. and Piterbarg, V.I. (1997): On the Distribution of the Maximum of a Gaussian Field with Constant Variance on a Smooth Manifold. *Theory of Probab. and Appl.*, **41**(2), 367-379.

Pickands, J. (1969): Upcrossing Probabilities for Stationary Gaussian Processes. *Transactions of the American Mathematical Society*, **145**, 51-73.

Piterbarg, V.I. and Stamatovich, S. (2001): On Maximum of Gaussian Non-centered Fields Indexed on Smooth Manifolds. In: *Asymptotic Methods in Probability and Statistics with Applications; Statistics for Industry and Technology,* Eds: N. Balakrishnan, I. A. Ibragimov, V. B. Nevzorov, Birkhäuser Boston, Boston, MA, pp. 189-203.

Qiao, W. and Polonik, W. (2014): Theoretical Analysis of Nonparametric Filament Estimation. *Submitted.*

Rosenblatt, M. (1976): On the Maximal Deviation of $k$-Dimensional Density Estimates. *The Annals of Probability*, **4**(6), 1009-1015.

Slepian, D. (1962): The one-sided barrier problem for the Gaussian noise. *Bell System Tech. J.*, **41**, 463-501.