Online Supplement for "Functional Models for Time-Varying Random Objects"

Paromita Dubey

University of Califonia, Davis

E-mail: pdubey@ucdavis.edu

Hans-Georg Müller

University of Califonia, Davis

Supplementary Materials

S1. Proofs

Proof of Proposition 1

It is clear that $C(s,t)$ is a symmetric function. To prove that it is nonnegative definite we need to show that for any positive integer k,

$$\sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j C(s_i, s_j) \geq 0,$$

for any $a_1, a_2, \ldots, a_k$ in $\mathbb{R}$ and $s_1, s_2, \ldots, s_k$ in $[0,1]$. Since d^2 is a semimetric of negative type, by Proposition 3 in Sejdinovic et al. (2013) there exists a Hilbert space $\mathcal{H}$ and an injective map $f : \Omega \to H$ such that $d^2(\omega_1, \omega_2) = \|f(\omega_1) - f(\omega_2)\|^2_{\mathcal{H}}$. We therefore have that for $x, x', y, y' \in \Omega$,

$$\begin{aligned}
&d^2(x,y') + d^2(x',y) - 2d^2(x,y) \\
=&\|f(x) - f(y')\|^2_{\mathcal{H}} + \|f(x') - f(y)\|^2_{\mathcal{H}} - 2\|f(x) - f(y)\|^2_{\mathcal{H}} \\
=&\|f(y')\|^2_{\mathcal{H}} + \|f(x')\|^2_{\mathcal{H}} - \|f(y)\|^2_{\mathcal{H}} - \|f(x)\|^2_{\mathcal{H}} + 4\langle f(x), f(y)\rangle_{\mathcal{H}} \\
&- 2\langle f(x), f(y')\rangle_{\mathcal{H}} - 2\langle f(x'), f(y)\rangle_{\mathcal{H}},
\end{aligned}$$

which implies that for i.i.d copies (X, Y) and (X', Y')

$$C(X,Y) = \frac{1}{4}E\left(4\langle f(X), f(Y)\rangle_{\mathcal{H}} - 2\langle f(X), f(Y')\rangle_{\mathcal{H}} - 2\langle f(X'), f(Y)\rangle_{\mathcal{H}}\right).$$

Let $H_i = f(X(s_i))$ and $H'_i = f(X'(s_i))$ for $i = 1, 2, \ldots, k$ where X' is an i.i.d copy of X. Then

$$C(s_i, s_j) = \frac{1}{4}E\left(4\langle H_i, H_j\rangle_{\mathcal{H}} - 2\langle H_i, H'_j\rangle_{\mathcal{H}} - 2\langle H'_i, H_j\rangle_{\mathcal{H}}\right),$$

which leads to

$$\sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j C(s_i, s_j)$$
$$= \frac{1}{4} E\left(4\left\langle \sum_{i=1}^{k} a_i H_i, \sum_{j=1}^{k} a_j H_j \right\rangle_{\mathcal{H}} - 2\left\langle \sum_{i=1}^{k} a_i H_i, \sum_{j=1}^{k} a_j H_j' \right\rangle_{\mathcal{H}} - 2\left\langle \sum_{i=1}^{k} a_i H_i', \sum_{i=1}^{k} a_j H_j \right\rangle_{\mathcal{H}}\right)$$
$$\geq \frac{1}{4} E\left(4\|\sum_{i=1}^{k} a_i H_i\|_{\mathcal{H}}^2\right) - 2E\left(\|\sum_{i=1}^{k} a_i H_i\|_{\mathcal{H}}\right) E\left(\|\sum_{j=1}^{k} a_j H_j'\|_{\mathcal{H}}\right)$$
$$- 2E\left(\|\sum_{i=1}^{k} a_i H_i'\|_{\mathcal{H}}\right) E\left(\|\sum_{j=1}^{k} a_j H_j\|_{\mathcal{H}}\right)$$
$$= \mathrm{Var}\left(\|\sum_{i=1}^{k} a_i H_i\|_{\mathcal{H}}\right).$$

The last step follows from the Cauchy-Schwarz inequality. This completes the proof.

Proof of Proposition 2

(a) For any $\gamma > 0$, by (I1) there exists $\delta > 0$ such that, whenever $|t_1 - t_2| < \delta$,

$$\sup_{\omega \in \Omega} |H(\omega, t_1) - H(\omega, t_2)| < \gamma.$$

For any partition $\mathcal{P}$ as described above such that $\epsilon_{\mathcal{P}} < \delta$ we find

$$\sup_{\omega \in \Omega} |I_{\mathcal{P}}(\omega) - I(\omega)|$$
$$= \sup_{\omega \in \Omega} |\sum_{j=0}^{k-1} H(\omega, t_j)\Delta_j - \int_0^1 H(\omega, t)dt|$$
$$= \sup_{\omega \in \Omega} |\sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} \{H(\omega, t_j) - H(\omega, t)\}dt|$$
$$= \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} \sup_{\omega \in \Omega} |H(\omega, t_j) - H(\omega, t)|dt \quad < \gamma,$$

which completes the proof for part (a).

(b) Observe that

$$
\begin{aligned}
&|I(\sum_{\mathcal{P},\oplus} S\phi) - I(\int_{\oplus} S\phi)| \\
&\leq |I(\sum_{\mathcal{P},\oplus} S\phi) - I_{\mathcal{P}}(\sum_{\mathcal{P},\oplus} S\phi) + I_{\mathcal{P}}(\sum_{\mathcal{P},\oplus} S\phi) - I(\int_{\oplus} S\phi)| \\
&\leq \sup_{\omega\in\Omega}|I(\omega) - I_{\mathcal{P}}(\omega)| + |\min_{\omega\in\Omega} I_{\mathcal{P}}(\omega) - \min_{\omega\in\Omega} I(\omega)| \\
&\leq 2\sup_{\omega\in\Omega}|I(\omega) - I_{\mathcal{P}}(\omega)|,
\end{aligned}
$$

and therefore by part (a), $|I(\sum_{\mathcal{P},\oplus} S\phi) - I(\int_{\oplus} S\phi)| \to 0$ as $\epsilon_{\mathcal{P}} \to 0$.

Now assume that $\lim_{\epsilon_{\mathcal{P}}\to 0} d(\sum_{\mathcal{P},\oplus} S\phi, \int_{\oplus} S\phi) \neq 0$. Then there must exist a sequence of partitions $\{\mathcal{P}_n\}$ and a $\gamma > 0$ such that $\epsilon_{\mathcal{P}_n} \to 0$ but $d(\sum_{\mathcal{P}_n,\oplus} S\phi, \int_{\oplus} S\phi) \geq \gamma$. For this sequence of partitions we observe that,

$$
|I(\sum_{\mathcal{P}_n,\oplus} S\phi) - I(\int_{\oplus} S\phi)| \geq |\inf_{d(\omega,\int_{\oplus} S\phi)>\gamma} I(\omega) - I(\int_{\oplus} S\phi)| > 0
$$

and therefore $\lim_{\epsilon_{\mathcal{P}_n}\to 0}|I(\sum_{\mathcal{P}_n,\oplus} S\phi) - I(\int_{\oplus} S\phi)| \geq |\inf_{d(\omega,\int_{\oplus} S\phi)>\gamma} I(\omega) - I(\int_{\oplus} S\phi)| > 0$ by (I2), which is a contradiction to part (a). Therefore the assumption that $\lim_{\epsilon_{\mathcal{P}}\to 0} d(\sum_{\mathcal{P},\oplus} S\phi, \int_{\oplus} S\phi) \neq 0$ cannot be true, which completes the proof for part(b).

(c) Let $\delta > 0$ be such that whenever $\epsilon_{\mathcal{P}} < \delta$, it holds that $d(\sum_{\mathcal{P},\oplus} S\phi, \int_{\oplus} S\phi) < \nu$. Assume that $\lim_{\epsilon_{\mathcal{P}}\to 0} h^{1/\beta}(\epsilon_{\mathcal{P}}) d(\sum_{\mathcal{P},\oplus} S\phi, \int_{\oplus} S\phi) \neq 0$. Then there exists a sequence of partitions $\{\mathcal{P}_n\}$ and a $\gamma > 0$ such that $\epsilon_{\mathcal{P}_n} < \delta$, while $d(\sum_{\mathcal{P}_n,\oplus} S\phi, \int_{\oplus} S\phi) \geq \frac{\gamma}{h^{1/\beta}(\epsilon_{\mathcal{P}})}$. For this sequence of partitions we observe that

$$
\begin{aligned}
&h(\epsilon_{\mathcal{P}_n})|I(\sum_{\mathcal{P}_n,\oplus} S\phi) - I(\int_{\oplus} S\phi)| \\
&\geq h(\epsilon_{\mathcal{P}_n})|\inf_{\frac{\gamma}{h^{1/\beta}(\epsilon_{\mathcal{P}_n})}\leq d(\omega,\int_{\oplus} S\phi)<\nu} I(\omega) - I(\int_{\oplus} S\phi)| \\
&\geq \frac{Ch(\epsilon_{\mathcal{P}_n})\gamma^{\beta}}{h(\epsilon_{\mathcal{P}_n})}
\end{aligned}
$$

by (I3). Therefore $\lim_{\epsilon_{\mathcal{P}_n}\to 0} h(\epsilon_{\mathcal{P}_n})|I(\sum_{\mathcal{P}_n,\oplus} S\phi) - I(\int_{\oplus} S\phi)| \geq C\gamma^{\beta} > 0$, which results in a contradiction and completes the proof for part (c).

Proof of Lemma 3

Consider $t \in [0,1]$ and a sequence $\{t_n\} \in [0,1]$ such that $t_n \to t$. We aim to show that $d(\mu_{\oplus}(t_n), \mu_{\oplus}(t)) \to 0$.

Observe that almost surely continuous sample curves on the compact interval $[0, 1]$ are uniformly continuous and since Ω is bounded, by bounded convergence, for all $t \in [0, 1]$ and sequences $\{t_n\} \in [0, 1]$ such that $t_n \to t$, there exists a $\delta > 0$ for every $\epsilon > 0$ such that whenever $|t_n - t| < \delta$, it holds that $E(d(X(t_n), X(t)) < \epsilon$ for all but finitely many n. For processes $E(d^2(\omega, X(t)))$,

$$|E(d^2(\omega, X(t_n))) - E(d^2(\omega, X(t)))| \leq 2D\ E(d(X(t_n), X(t)),$$

and therefore, given any $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|t_n - t| < \delta$, one has $\sup_{\omega\in\Omega}|E(d^2(\omega, X(t_n))) - E(d^2(\omega, X(t)))| < \varepsilon$. This implies that

$$\begin{aligned}
&|E(d^2(\mu_\oplus(t_n), X(t))) - E(d^2(\mu_\oplus(t), X(t)))| \\
&\leq |E(d^2(\mu_\oplus(t_n), X(t))) - E(d^2(\mu_\oplus(t_n), X(t_n))) \\
&\qquad + E(d^2(\mu_\oplus(t_n), X(t_n))) - E(d^2(\mu_\oplus(t), X(t)))| \\
&\leq \sup_{\omega\in\Omega}|E(d^2(\omega, X(t_n))) - E(d^2(\omega, X(t)))| + |\min_{\omega\in\Omega} E(d^2(\omega, X(t_n))) - \min_{\omega\in\Omega} E(d^2(\omega, X(t)))| \\
&\leq 2 \sup_{\omega\in\Omega}|E(d^2(\omega, X(t_n))) - E(d^2(\omega, X(t)))| \quad < 2\epsilon.
\end{aligned}$$

Assume $d(\mu_\oplus(t_n), \mu_\oplus(t)) \nrightarrow 0$. Then one can find an $\eta > 0$ such that for any $\delta > 0$, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ for which $|t_{n_k} - t| < \delta$ but $d(\mu_\oplus(t_{n_k}), \mu_\oplus(t)) \geq \eta$. Then by (A3),

$$\begin{aligned}
&|E(d^2(\mu_\oplus(t_{n_k}), X(t))) - E(d^2(\mu_\oplus(t), X(t)))| \\
&\geq |\inf_{d(\omega,\mu_\oplus(t))>\eta} E(d^2(\omega, X(t))) - E(d^2(\mu_\oplus(t), X(t)))| \\
&> 0.
\end{aligned}$$

This leads to a contradiction for $\epsilon = |\inf_{d(\omega,\mu_\oplus(t))>\eta} E(d^2(\omega, X(t))) - E(d^2(\mu_\oplus(t), X(t)))|/2$, thus completing the proof.

Proof of Theorem 1

Denoting the usual L^2 norm by $||\cdot||_2$, observe that for $s_1, t_1, s_2, t_2 \in [0, 1]$, one has

$$\begin{aligned}
&|f_{s_1,t_1}(x, y) - f_{s_2,t_2}(x, y)| \\
&\leq 4M\{d(x(s_1), x(s_2)) + d(x(t_1), x(t_2)) + d(y(s_1), y(s_2)) + d(y(t_1), y(t_2))\} \\
&\leq 4M\ (G(x) + G(y))\ (|s_1 - s_2|^\alpha + |t_1 - t_2|^\alpha),
\end{aligned}$$

implying

$$||f_{s_1,t_1} - f_{s_2,t_2}||_2 \leq 8M||G||_2(|s_1 - s_2|^\alpha + |t_1 - t_2|^\alpha).$$

Observe that for any $0 < u < 1$, if we take $|s_1 - s_2| < \left(\frac{u}{16}\right)^{\frac{1}{\alpha}}$ and $|t_1 - t_2| < \left(\frac{u}{16}\right)^{\frac{1}{\alpha}}$, then $||f_{s_1,t_1}(X) - f_{s_2,t_2}(Y)||_2 \leq Mu||G||_2$. Therefore, with $s_1, s_2, \ldots, s_K$ and $t_1, t_2, \ldots, t_L$ forming $\left(\frac{u}{4}\right)^{\frac{1}{\alpha}}$-nets for $[0, 1]$ with metric $|\cdot|$, the brackets $[f_{s_i,t_j} \pm Mu||G||_2]$ cover the function class $\mathcal{F}$ (Van der Vaart and Wellner, 1996) and are of length $2Mu||G||_2$. This implies

$$N_{[]}(2Mu||G||_2, \mathcal{F}, L^2(P \otimes P)) \leq N\left(\left(\frac{u}{4}\right)^{\frac{1}{\alpha}}, [0,1], |\cdot|\right)^2,$$

where $N_{[]}(\varepsilon, \mathcal{F}, L^2(P))$ is the bracketing number, which is the minimum number of ε-brackets needed to cover $\mathcal{F}$, where an ε-bracket is composed of pairs of functions $[l, u]$ such that $||l - u||_2 < \varepsilon$, and N is the covering number. Hence for any $\epsilon > 0$, for some constant $K > 0$,

$$N_{[]}(\epsilon, \mathcal{F}, L^2(P \otimes P)) \leq K\epsilon^{-2/\alpha} < \infty.$$

The result now follows from Theorem 4.10 of Arcones and Giné (1993), observing

$$\begin{aligned}
&\int_0^1 \sqrt{\log N_{[]}(\epsilon, \mathcal{F}, L^2(P \otimes P))}d\varepsilon \\
&\leq \varepsilon\sqrt{\log K} + \int_0^1 \sqrt{-\frac{2}{\alpha}\log \varepsilon d\varepsilon} \\
&= \varepsilon\sqrt{\log K} + \sqrt{\frac{2}{\alpha}}\Gamma\left(\frac{3}{2}\right) < \infty.
\end{aligned}$$

Proof of Corollary 1

For any fixed j, Lemma 4.3 in Bosq (2000) gives $|\hat{\lambda}_j - \lambda_j| \leq \sup_{s,t\in[0,1]} \left|\hat{C}(s,t) - C(s,t)\right|$. Uniform mapping then implies

$$|\hat{\lambda}_j - \lambda_j| = O_P(1/\sqrt{n}).$$

Under assumption (A5), $\sup_{s\in[0,1]} \left|\hat{\phi}_j(s) - \phi_j(s)\right| \leq 2\sqrt{2}\delta_j^{-1} \sup_{s,t\in[0,1]} \left|\hat{C}(s,t) - C(s,t)\right|$, and therefore

$$\sup_{s\in[0,1]} \left|\hat{\phi}_j(s) - \phi_j(s)\right| = O_P(1/\sqrt{n}),$$

which completes the proof.

Proof of Theorem 2

Since

$$\left|\int_0^1 \hat{\phi}(t)dt - \int_0^1 \phi(t)dt\right| \leq \sup_{s\in[0,1]} \left|\hat{\phi}(s) - \phi(s)\right| = O_P(1/\sqrt{n}), \tag{1}$$

for all sufficiently large n, $\left|\int_0^1 \hat\phi(t)dt\right| \geq \left|\int_0^1 \phi(t)dt\right|/2$, and since $\left|\int_0^1 \phi(t)dt\right| \neq 0$,

$$\begin{aligned}
&\sup_{s\in[0,1]} |\hat\phi^*(s) - \phi^*(s)| \\
&\leq 2\frac{\sup_{s\in[0,1]}\left|\hat\phi(s)\right|\left|\int_0^1 \hat\phi(t)dt - \int_0^1 \phi(t)dt\right| + \left|\int_0^1 \hat\phi(t)dt\right|\sup_{s\in[0,1]}\left|\hat\phi(s) - \phi(s)\right|}{\left|\int_0^1 \phi(t)dt\right|^2} \\
&= O_P(1/\sqrt{n}).
\end{aligned}$$

As a direct consequence,

$$\begin{aligned}
&\sup_{\omega\in\Omega}\left|\int_0^1 d^2(\omega, X_i(t))\hat\phi^*(t)dt - \int_0^1 d^2(\omega, X_i(t))\phi^*(t)dt\right| \\
&\leq M^2 \sup_{s\in[0,1]} |\hat\phi^*(s) - \phi^*(s)| = O_P(1/\sqrt{n}), \qquad (2)
\end{aligned}$$

whence

$$\begin{aligned}
&P\left(d(\hat\psi_\oplus^{ik}, \psi_\oplus^{ik}) > \epsilon\right) \\
&\leq P\left(\left|\int_0^1 d^2(\hat\psi_\oplus^{ik}, X_i(t))\phi^*(t)dt - \int_0^1 d^2(\psi_\oplus^{ik}, X_i(t))\phi^*(t)dt\right| > c_\varepsilon\right) \qquad (3)\\
&\leq P\left(\sup_{\omega\in\Omega}\left|\int_0^1 d^2(\omega, X_i(t))\hat\phi^*(t)dt - \int_0^1 d^2(\omega, X_i(t))\phi^*(t)dt\right| > \frac{c_\varepsilon}{2}\right), \qquad (4)
\end{aligned}$$

which implies that $d(\hat\psi_\oplus^{ik}, \psi_\oplus^{ik}) = o_P(1)$ by equation (2). Here (4) follows from (3) using the fact that

$$\begin{aligned}
&\left|\int_0^1 d^2(\hat\psi_\oplus^{ik}, X_i(t))\phi^*(t)dt - \int_0^1 d^2(\psi_\oplus^{ik}, X_i(t))\phi^*(t)dt\right| \\
&\leq \left|\int_0^1 d^2(\hat\psi_\oplus^{ik}, X_i(t))\left(\phi^*(t) - \hat\phi^*(t)\right)dt\right| \\
&+ \left|\inf_{\omega\in\Omega}\int_0^1 d^2(\omega, X_i(t))\hat\phi^*(t)dt - \inf_{\omega\in\Omega}\int_0^1 d^2(\omega, X_i(t))\phi^*(t)dt\right| \\
&\leq 2\sup_{\omega\in\Omega}\left|\int_0^1 d^2(\omega, X_i(t))\hat\phi^*(t)dt - \int_0^1 d^2(\omega, X_i(t))\phi^*(t)dt\right|.
\end{aligned}$$

From assumption (A7),

$$\begin{aligned}
&P\left(n^{1/(2\beta_1)}d(\hat\psi_\oplus^{ik}, \psi_\oplus^{ik}) > 2^L\right) \\
&\leq P\left(\frac{2^L}{n^{1/(2\beta_1)}} < d(\hat\psi_\oplus^{ik}, \psi_\oplus^{ik}) < \nu'\right) + P\left(d(\hat\psi_\oplus^{ik}, \psi_\oplus^{ik}) \geq \nu'\right) \\
&= P\left(\sup_{\omega\in\Omega}\left|\int_0^1 d^2(\omega, X_i(t))\hat\phi^*(t)dt - \int_0^1 d^2(\omega, X_i(t))\phi^*(t)dt\right| > \frac{2^{\beta_1 L}}{n^{1/2}}\right) + P\left(d(\hat\psi_\oplus^{ik}, \psi_\oplus^{ik}) \geq \nu'\right) \\
&\leq P\left(\sqrt{n}\sup_{s\in[0,1]} |\hat\phi^*(s) - \phi^*(s)| > \frac{2^{\beta_1 L}}{M^2}\right) + P\left(d(\hat\psi_\oplus^{ik}, \psi_\oplus^{ik}) \geq \nu'\right).
\end{aligned}$$

Therefore $P\left(n^{1/2\beta_1} d(\hat{\psi}_{\oplus}^{ik}, \psi_{\oplus}^{ik}) > 2^L\right)$ can be made as small as possible by choosing sufficiently large n and L, using equation (2) and the fact that $d(\hat{\psi}_{\oplus}^{ik}, \psi_{\oplus}^{ik}) = o_P(1)$, thus completing the proof.

Proof of Proposition 4

By Theorem 1.5.4 in Van der Vaart and Wellner (1996), it suffices to show asymptotic equicontinuity of the processes $Z_n(s) = d(\hat{\mu}_{\oplus}(s), \mu_{\oplus}(s))$, i.e. for any $\theta > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\left(\sup_{|s-t|<\delta} |Z_n(s) - Z_n(t)| > 2\theta\right) = 0, \tag{5}$$

in addition to the pointwise convergence of $Z_n(s)$, i.e. for all $s \in [0,1]$ it holds that

$$Z_n(s) = o_P(1). \tag{6}$$

We observe that equation (6) follows from Theorem 1 in Petersen and Müller (2019a). To establish equation (5), by Lemma 3 and as $|Z_n(s) - Z_n(t)| \le d(\mu_{\oplus}(s), \mu_{\oplus}(t)) + d(\hat{\mu}_{\oplus}(s), \hat{\mu}_{\oplus}(t))$, it suffices to show that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\left(\sup_{|s-t|<\delta} d(\hat{\mu}_{\oplus}((s), \hat{\mu}_{\oplus}((t)) > \theta\right) = 0. \tag{7}$$

To show equation (7), suppose that $d(\hat{\mu}_{\oplus}(s), \hat{\mu}_{\oplus}(t)) > \theta$ with $s, t \in [0,1]$.

Step 1. Since the functions are continuous and the domain is compact, it holds that almost surely $\sup_{|s-t|<\delta} d(X(t), X(s)) \to 0$ as $\delta \to 0$. By the boundedness of the metric and dominated convergence,

$$\lim_{\delta \to 0} E(\sup_{|s-t|<\delta} d(X(t)), X(s))) \to 0. \tag{8}$$

Now (8) implies that for any $a > 0$,

$$\begin{aligned} &P\left(\sup_{|s-t|<\delta} \sup_{\omega \in \Omega} \left| \frac{1}{n}\sum_{i=1}^n d^2(X_i(t), \omega) - \frac{1}{n}\sum_{i=1}^n d^2(X_i(s), \omega)\right| > a\right) \\ &\le \frac{2M \; E\left(\sup_{|s-t|<\delta} \frac{1}{n}\sum_{i=1}^n d(X_i(t), X_i(s))\right)}{a} \\ &\le \frac{2M \; E(\sup_{|s-t|<\delta} d(X(t)), X(s)))}{a} \to 0 \quad \text{as } \delta \to 0, \end{aligned}$$

where $M = \text{diam}(\Omega)$.

Step 2. We observe that

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\hat{\mu}_\oplus(s)) - \frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\hat{\mu}_\oplus(t))\right| \\
&\le \left|\frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\hat{\mu}_\oplus(s)) - \frac{1}{n}\sum_{i=1}^{n} d^2(X_i(s),\hat{\mu}_\oplus(s))\right| \\
&\qquad + \left|\frac{1}{n}\sum_{i=1}^{n} d^2(X_i(s),\hat{\mu}_\oplus(s)) - \frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\hat{\mu}_\oplus(t))\right| \\
&\le 2\sup_{\omega\in\Omega}\left|\frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\omega) - \frac{1}{n}\sum_{i=1}^{n} d^2(X_i(s),\omega)\right|.
\end{aligned}$$

Step 3. Now we find

$$\begin{aligned}
&P\left(\sup_{|s-t|<\delta} d(\hat{\mu}_\oplus(s),\hat{\mu}_\oplus(t)) > \theta\right) \\
&\le P\left(B\cap A_n\right) + P\left(B\cap A_n^C\right) \qquad (9)\\
&\le P\left(\sup_{|s-t|<\delta} 2\sup_{\omega\in\Omega}\left|\frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\omega) - \frac{1}{n}\sum_{i=1}^{n} d^2(X_i(s),\omega)\right| \ge \tau(S)\right) + P\left(B\cap A_n^C\right).
\end{aligned}$$

where $A_n = \{\sup_{|s-t|<\delta}\left|\frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\hat{\mu}_\oplus(s)) - \frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\hat{\mu}_\oplus(t))\right| \ge \tau(S)\}$, $B = \{\sup_{|s-t|<\delta} d(\hat{\mu}_\oplus(s),\hat{\mu}_\oplus(t)) > S\}$ in (9) with $\tau(S)$ as defined in (A8). The last step follows from (9) using Step 2. From Step 1, choosing $a = \frac{\tau(S)}{2}$,

$$\lim_{\delta\to 0} P\left(\sup_{|s-t|<\delta} 2\sup_{\omega\in\Omega}\left|\frac{1}{n}\sum_{i=1}^{n} d^2(X_i(t),\omega) - \frac{1}{n}\sum_{i=1}^{n} d^2(X_i(s),\omega)\right| \ge \tau(S)\right) = 0$$

for any n, and from (A1), $\lim_{n\to\infty} P\left(B\cap A_n^C\right) = 0$. This completes the proof.

Proof of Lemma 1

Define functions $f_{\omega,s}(x) = d^2(x(s),\omega(s)) - d^2(x(s),\mu_\oplus(s))$. We find

$$\begin{aligned}
&|f_{\omega_1,s_1}(x) - f_{\omega_2,s_2}(x)| \\
&\le |f_{\omega_1,s_1}(x) - f_{\omega_1,s_2}(x)| + |f_{\omega_1,s_2}(x) - f_{\omega_2,s_2}(x)| \\
&\le 2M\left(2d(x(s_1),x(s_2)) + d(\mu_\oplus(s_1),\mu_\oplus(s_2)) + d(\omega_1(s_1),\omega_1(s_2)) + d(\omega_1(s_2),\omega_2(s_2))\right).
\end{aligned}$$

Note that $d(\omega_1(s_2),\omega_2(s_2)) \le d(\omega_1(s_2),\mu_2(s_2)) + d(\omega_1(s_2),\mu_2(s_2))$. By assumptions (A4) and (A9), it holds that almost surely,

$$\begin{aligned}
&|f_{\omega_1,s_1}(X) - f_{\omega_2,s_2}(X)| \le 4M\left[G(X)|s_1-s_2|^\alpha + H_\delta|s_1-s_2|^{\nu_\delta}\right] \\
&+ 4M\left[d(\omega_1(s_2),\mu_2(s_2)) + d(\omega_1(s_2),\mu_2(s_2))\right],
\end{aligned}$$

which implies that

$$\begin{aligned}&||f_{\omega_1,s_1} - f_{\omega_2,s_2}||_2\\&\quad \leq 4M\left[||G||_2|s_1-s_2|^{\alpha} + H_\delta|s_1-s_2|^{\nu_\delta} + d(\omega_1(s_2),\mu_2(s_2)) + d(\omega_1(s_2),\mu_2(s_2))\right].\end{aligned}$$

It follows that for some $0 < u < 1$, if we take $|s_1 - s_2| < \left(\frac{u}{4}\right)^{\frac{1}{V}}$ with $V = \min(\alpha, \nu_\delta)$ and ω_1, ω_2 such that $d_\infty(\omega_1, \mu) < \frac{u}{8}$ and $d_\infty(\omega_2, \mu) < \frac{u}{8}$, then $||f_{\omega_1,s_1} - f_{\omega_2,s_2}||_2 < Mu(||G||_2 + H_\delta + 1)$. Therefore if $s_1, s_2, \ldots, s_K$ is a $\left(\frac{u}{4}\right)^{\frac{1}{V}}$-net for $[0,1]$ with metric $|\cdot|$ and $\omega_1, \omega_2, \ldots, \omega_L$ is a $\frac{u}{8}$-net for $B_\delta(\mu(\cdot))$ with metric d_∞, the brackets $[f_{s_i,\omega_j} \pm Mu(||G||_2 + H_\delta + 1)]$ cover the function class $\mathcal{F}_\delta$ and are of length $2Mu(||G||_2 + H_\delta + 1)$. We conclude that

$$N_{[]}(2Mu(||G||_2 + H_\delta + 1), \mathcal{F}_\delta, L^2(P)) \leq N\left(\left(\frac{u}{4}\right)^{\frac{1}{V}}, [0,1], |\cdot|\right) N\left(\frac{u}{8}, B_\delta(\mu_\oplus(\cdot)), d_\infty\right).$$

Applying Van der Vaart and Wellner (1996) (page 84), for any function class $\mathcal{F}$ and for any r,

$$N(\varepsilon, \mathcal{F}, L^r(P)) \leq N_{[]}(2\varepsilon, \mathcal{F}, L^r(P)),$$

so that for appropriate constants $K_1, K_2, C > 0$,

$$\begin{aligned}&\log N(2M\delta\varepsilon, \mathcal{F}_\delta, L^2(P))\\&\leq \log N\left(K_1(\varepsilon\delta)^{1/V}, [0,1], |\cdot|\right) + \log N\left(K_2\varepsilon\delta, B_\delta(\mu_\oplus(\cdot)), d_\infty\right)\\&\leq \log\left(C\left(\frac{1}{\varepsilon\delta}\right)^{1/V}\right) + \log N\left(K_2\varepsilon\delta, B_\delta(\mu_\oplus(\cdot)), d_\infty\right).\end{aligned}$$

Observe that $\log N\left(K_2\varepsilon\delta, B_\delta(\mu_\oplus(\cdot)), d_\infty\right) \leq \sup_{s\in[0,1]} \log N\left(K_2\varepsilon\delta, B_\delta(\mu_\oplus(s)), d\right)$ because $d_\infty(\omega_1, \omega_2) = \sup_{s\in[0,1]} d(\omega_1(s), \omega_2(s))$ and $d(\omega_1(s), \omega_2(s))$ is a uniformly continuous function in s so that the supremum is attained. Therefore, $d_\infty(\omega_1, \omega_2) = d(\omega_1(s^*), \omega_2(s^*))$ for some $s^* \in [0,1]$. Finally we observe that

$$\begin{aligned}&\int_0^1 \sqrt{1 + \log N(\varepsilon||F||_2, \mathcal{F}_\delta, L^2(P))}d\varepsilon\\&= \int_0^1 \sqrt{1 + \log N(2M\delta\varepsilon, \mathcal{F}_\delta, L^2(P))}d\varepsilon\\&\leq \sqrt{\log(C)} + \int_0^1 \sqrt{-\frac{1}{V}\log(\varepsilon\delta)}d\varepsilon + \int_0^1 \sup_{s\in[0,1]} \sqrt{\log N\left(K_2\varepsilon\delta, B_\delta(\mu_\oplus(s)), d\right)}d\varepsilon\\&\leq \sqrt{\log(C)} + \frac{1}{\sqrt{V}}\int_0^1 \sqrt{-\log(\varepsilon\delta)}d\varepsilon + \int_0^1 \sup_{s\in[0,1]} \sqrt{\log N\left(K_2\varepsilon\delta, B_\delta(\mu_\oplus(s)), d\right)}d\varepsilon.\end{aligned}$$

Assumption (A10) then implies $J_{[]}(1, \mathcal{F}_\delta, L^2(P)) = O(\sqrt{-\log\delta})$ as $\delta \to 0$, which completes the proof.

Proof of Theorem 3

For a sequence $\{q_n\}$ define the sets

$$S_{j,n}(x) = \{\omega(\cdot) : 2^{j-1} < q_n d_\infty^{\beta_2/2}(\omega, \mu_\oplus) \le 2^j\}.$$

Choose $\alpha > 0$ to satisfy (A11) and also small enough such that (A3) and (A4) hold for all $\delta < \alpha$ and choose $\tilde{\alpha} = \alpha^{\beta_2/2}$. For any integer L,

$$\begin{aligned} &P\left(q_n d_\infty^{\beta_2/2}(\hat{\mu}_\oplus, \mu_\oplus) > 2^L\right) \\ &\le P\left(d_\infty(\hat{\mu}_\oplus, \mu_\oplus) \ge \alpha\right) + \sum_{j \ge L, 2^j \le q_n \tilde{\alpha}} P\left(\hat{\mu}_\oplus \in S_{j,n}\right) \\ &\le P\left(d_\infty(\hat{\mu}_\oplus, \mu_\oplus) \ge \alpha\right) + \sum_{j \ge L, 2^j \le q_n \tilde{\alpha}} P\left(\sup_{\omega \in S_{j,n}} |V_n(\omega, s) - V(\omega, s)| \ge D\frac{2^{2(j-1)}}{q_n^2},\right) \end{aligned} \tag{10}$$

where (10) follows by observing

$$\sup_{\omega \in S_{j,n}} |V_n(\omega, s) - V(\omega, s)| \ge \left| \inf_{\omega \in S_{j,n}} V_n(\omega, s) - \inf_{\omega \in S_{j,n}} V(\omega, s) \right| \ge D\frac{2^{2(j-1)}}{q_n^2}.$$

The first term in (10) goes to zero by Proposition 4 and for each j in the second term it holds that $d_\infty(\omega, \mu_\oplus) \le \alpha$. By Lemma 1, $J_{[]}(1, \mathcal{F}_\delta, L^2(P)) = O(\sqrt{\log 1/\delta})$, and therefore is bounded above by $J\sqrt{\log 1/\delta}$ for all small enough $\delta > 0$, where $J > 0$ is a constant. Using equation (14), Lemma 1 and the Markov inequality, the second term is upper bounded up to a constant by

$$\sum_{j \ge L, 2^j \le q_n \tilde{\alpha}} \frac{2MJ2^j}{n} \sqrt{\log n / 2^{j+1}} \frac{q_n^2}{D 2^{2(j-1)}}. \tag{11}$$

Since $\sqrt{\log n/2^{j+1}}$ is dominated by $\sqrt{\log n/2}$, setting $q_n = \frac{\sqrt{n}}{(\log n)^{1/4}}$, the series in (11) is upper bounded by $\frac{8MJ}{D} \sum_{j \ge L, 2^j \le q_n \tilde{\alpha}} \frac{1}{2^j}$, which converges and can be made sufficiently small by choosing L and n large. This proves the desired result that $d_\infty(\hat{\mu}_\oplus, \mu_\oplus) = O_P(q_n^{-2/\beta_2}) = O_P\left(\left(\frac{n}{\sqrt{\log n}}\right)^{-1/\beta_2}\right)$.

Proof of Corollary 2

Observing that

$$\begin{aligned} &\left|\hat{\beta}_{ik} - \beta_{ik}\right| \\ &\le \left|\int_0^1 d(X_i(t), \hat{\mu}_\oplus(t)) \left(\hat{\phi}_k(t) - \phi_k(t)\right) dt\right| + \left|\int_0^1 \phi(t) \left(d(X_i(t), \hat{\mu}_\oplus(t)) - d(X_i(t), \mu_\oplus(t))\right) dt\right| \\ &\le M \sup_{s \in [0,1]} \left|\hat{\phi}_k(s) - \phi_k(s)\right| + \int_0^1 |\phi(t)| dt \sup_{s \in [0,1]} d(\hat{\mu}_\oplus(s), \mu_\oplus(s)), \end{aligned}$$

the result follows from Corollary 1 and Theorem 3.

S2. Comparison of Metric Covariance with Distance Covariance

For two random variables X and Y with marginal probability measures P_X and P_Y and joint probability measure P_{XY}, testing for probabilistic dependence corresponds to testing

$$H_0 : P_{XY} = P_X P_Y \quad \text{versus} \quad H_1 : P_{XY} \neq P_X P_Y. \tag{12}$$

Implementation of these tests is usually based on a metric in the space of probability measures. As shown in Lyons (2013), distance correlation (Székely et al., 2007; Székely and Rizzo, 2017) provides a suitable metric for this purpose, provided X and Y take values in metric spaces which are of "strong negative type" (Lyons, 2013) and include Euclidean spaces and separable Hilbert spaces. Then independence of X and Y is equivalent to the distance correlation being 0. While it is often of interest to determine whether distance correlation is zero, which then implies independence of X and Y, the magnitude of distance correlation if not zero is hard to interpret. This fact is emphasized for example in Jakobsen (2017) (page 61) where distance covariance is characterized to be useful to test for independence (12) but much less so to measure degree of dependence between random variables X and Y in general metric spaces.

As a concrete example, we compare distance correlation/covariance with metric correlation/covariance for the case of distribution spaces with the Wasserstein metric. Writing Q_1, Q_2 for the quantile function functions corresponding to distributions F_1, F_2, the distance covariance $\mathrm{dCov}(F_1, F_2)$ between F_1 and F_2 is found to correspond to

$$\begin{aligned}
\mathrm{dCov}(F_1, F_2) \;=\; & E\left[\int_0^1 \{Q_1(t) - Q_1'(t)\}^2 dt \int_0^1 \{Q_2(t) - Q_2'(t)\}^2 dt\right]^{1/2} \\
& + E\left[\int_0^1 \{Q_1(t) - Q_1'(t)\}^2 dt\right]^{1/2} E\left[\int_0^1 \{Q_2(t) - Q_2'(t)\}^2 dt\right]^{1/2} \\
& - 2E\left[\int_0^1 \{Q_1(t) - Q_1'(t)\}^2 dt \int_0^1 \{Q_2(t) - \tilde{Q}_2(t)\}^2 dt\right]^{1/2},
\end{aligned}$$

where F_1' is an independent copy of F_1 and F_2', $\tilde{F}_2$ are two independent copies of F_2. This expression for distance covariance is rather unintuitive, and it is hard to interpret as a measure for the strength of covariation between F_1 and F_2.

A numerical comparison provides further illumination. We implemented distance covariance as an alternative covariance/correlation for functional random objects in a simulation study, where we compared the utility of the proposed metric covariance with that of distance covariance (Székely and Rizzo, 2017; Lyons, 2013) for carrying out FPCA of regular scalar-valued functional data. We consider a simple setting as in classical FDA where the time-varying random objects $X(s)$ are real valued for $s \in [0, 1]$. In the following,

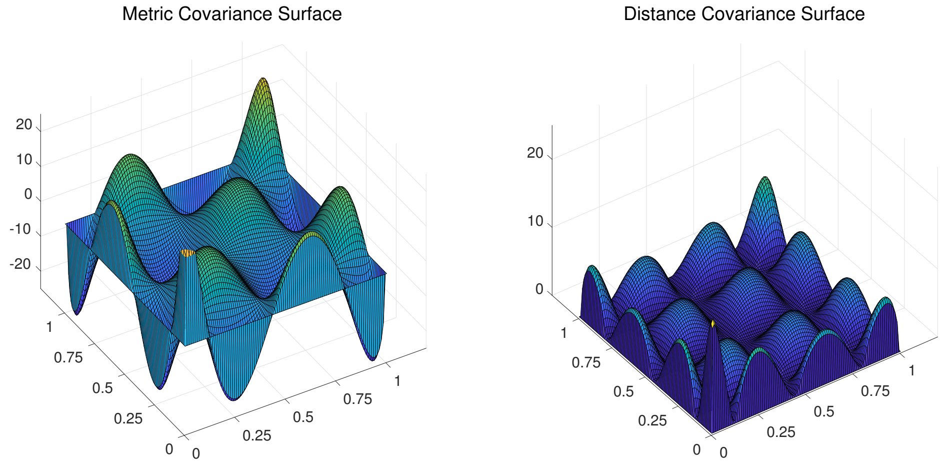


Fig. 1: Metric covariance surface (left) and distance covariance surface (right) for functional data generated according to model in (13).

we take U_i, V_i and Y_i to be distributed as $N(0,3)$, $N(0,1)$ and $N(0,0.25)$, respectively. As described in the simulation setting for time varying networks in Section 5.2, we take $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ to be orthonormal polynomials derived from the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (Totik, 2005), which are classical orthogonal polynomials for $\alpha, \beta > 1$. The expressions for $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ are

$$\begin{aligned}
\phi_1(t) &= \frac{(P_3^{(1,0.25)}(2t-1))t^{0.25}(1-t)^{0.5}}{[\int_0^1 (P_3^{(1,0.5)}(2t-1))^2 t^{0.5}(1-t)dt]^{1/2}} \\
\phi_2(t) &= \frac{(P_4^{(1,0.25)}(2t-1))t^{0.25}(1-t)^{0.5}}{[\int_0^1 (P_4^{(1,0.5)}(2t-1))^2 t^{0.5}(1-t)dt]^{1/2}} \\
\phi_3(t) &= \frac{P_5^{(1,0.5)}(2t-1))t^{0.25}(1-t)^{0.5}}{[\int_0^1 (P_5^{(1,0.5)}(2t-1))^2 t^{0.5}(1-t)dt]^{1/2}}.
\end{aligned}$$

We generated 1000 i.i.d. realizations $X_i(s)$ as follows on a fine grid of $[0,1]$,

$$X_i(s) = U_i\phi_1(t) + V_i\phi_2(t) + Y_i\phi_3(t). \tag{13}$$

It is clear from the construction of X_i that in model (13), the first, second and third eigenfunctions are given by $\phi_1(t), \phi_2(t)$ and $\phi_3(t)$ respectively. We evaluated the estimated metric covariance and distance covariance surfaces on a fine grid, which led to the surfaces depicted in Figure 1, and then obtained the first three eigenfunctions using these surfaces as covariance kernels. The resulting eigenfunctions are presented in Figure 2. As can be seen from Figures 1 and 2, metric covariance delivers the eigenfunctions that one would expect for functional principal component analysis, while distance covariance as an alter-

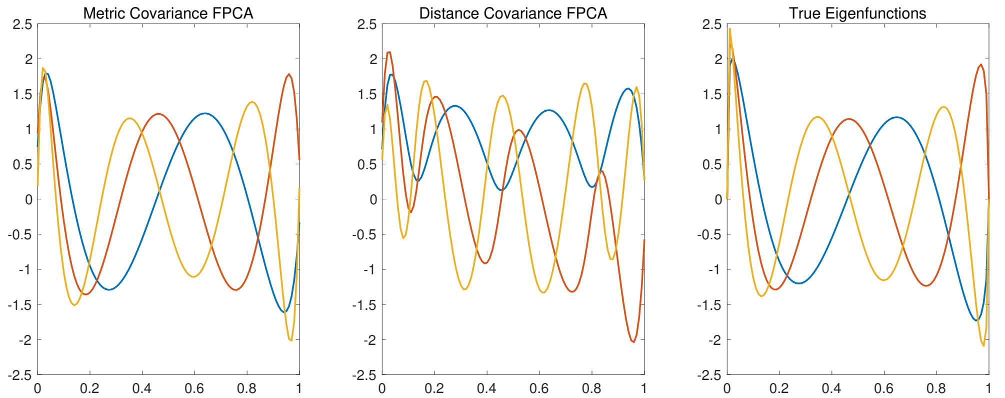


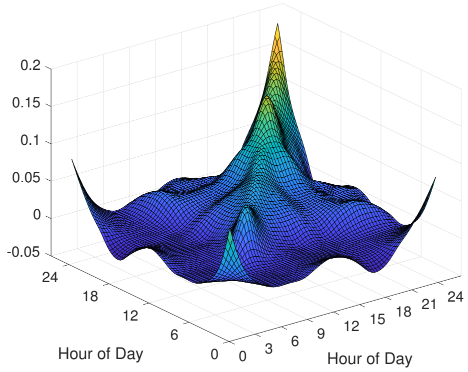
Fig. 2: Eigenfunctions obtained from using metric covariance based (left panel) and distance covariance based (middle panel) kernel for simulated functional data, generated according to model (13). Also shown are the true underlying eigenfunctions (right panel). The blue curves correspond to the first, the red curves to the second and the yellow curves to the third eigenfunction.

nate notion of covariance leads to seemingly arbitrary and uninterpretable eigenfunctions, so is clearly not suitable in this context.

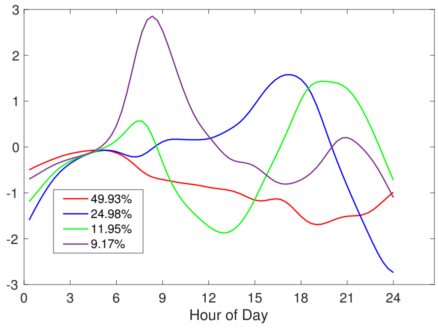
In classical FPCA one aims to identify dominant modes of variation of functional data that are derived from the eigenfunctions of the auto-covariance operator. The interpretation of these modes of variation provides valuable insights in many applications, and this is why interpretability of the eigenfunctions is important. Another major goal is to decompose the variation of functional data in a parsimonious and interpretable way into orthogonal directions. As illustrated in the simulation above, the lack of clear interpretation of the eigenfunctions associated with the distance covariance operator is a big hurdle for this program. When using distance correlation, the corresponding distance variance and also the total variation have an unintuitive and complex form that makes distance covariance rather unsuitable for our purposes. In contrast, the proposed metric covariance (5) works well for quantifying the variation and co-variation of random objects. It gives rise to the total variation measure (6) for functional random objects and emerges as a bona fide extension of the proven and successful FPCA for scalar-valued functional data.

S3. Metric auto-covariance surfaces and eigenfunctions for New York taxi data

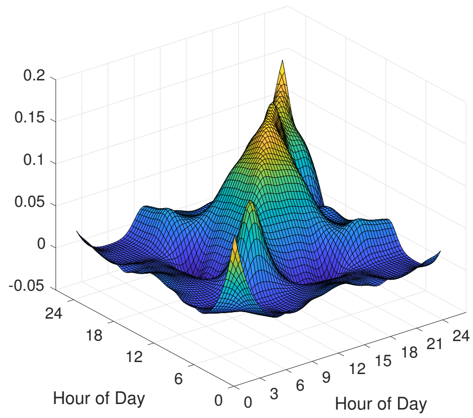
We repeated the analysis of the time-varying networks generated by the New York taxi data separately for three groups of days, namely the weekdays Monday-Thursday (group 1), Fridays and weekends (group 2) and holidays (group 3). The results are visualized in Figure 3. This figure clearly indicates that the metric covariance structure and the eigenfunctions differ across the groups. The Fréchet integrals for the dominant eigenfunctions reveal different aspects of variation, both within and between daily networks in three groups and are presented in the movies "week.mov", "friday.mov" and "Hol.mov" which are included in the supplementary materials. In the movie frames, the top left panels correspond to the FPCs for the first eigenfunction, the top right panels to those of the second, the bottom left panels to those of the third and the bottom right panels to those of the fourth eigenfunction. The edge weights in the graphs are proportional to their line widths.



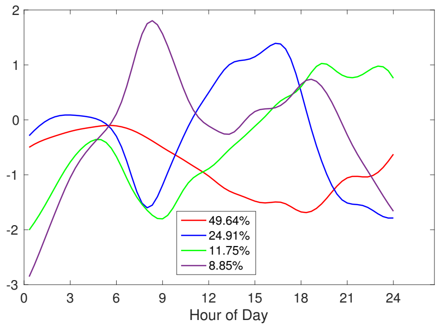
(a) Mondays-Thursdays



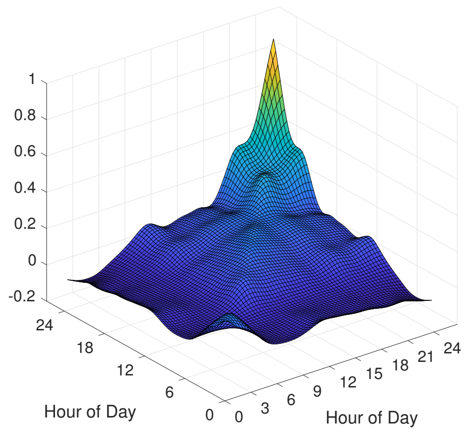
(b) Eigenfunctions



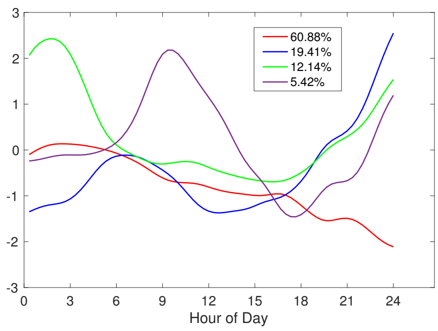
(c) Fridays



(d) Eigenfunctions



(e) Weekends and Holidays



(f) Eigenfunctions

Fig. 3: Estimated metric auto-covariance surface (10) (left) and associated eigenfunctions (right), for the New York taxi data, viewed as time-varying networks, separated by groups of days.

S4. Additional Visualization for the World Trade Data analysis

We selected USA, Saudi Arabia, Hong Kong and Thailand and display their time evolving inter-commodity trade correlations as obtained from the fitted model for the years 1970, 1982, 1992, 1999 (bottom to top) in Figure 4.

We also computed the Fréchet integral covariance matrices for the first four eigenfunctions. For visualization of commodities trade similarities these Fréchet integral covariance matrices were converted to correlation matrices that can be viewed in the movie "trade.mov", available in the online supplement. In the movie frames, the FPCs for the first, second, third and fourth eigenfunctions are at the top left, top right, bottom left and bottom right, respectively.

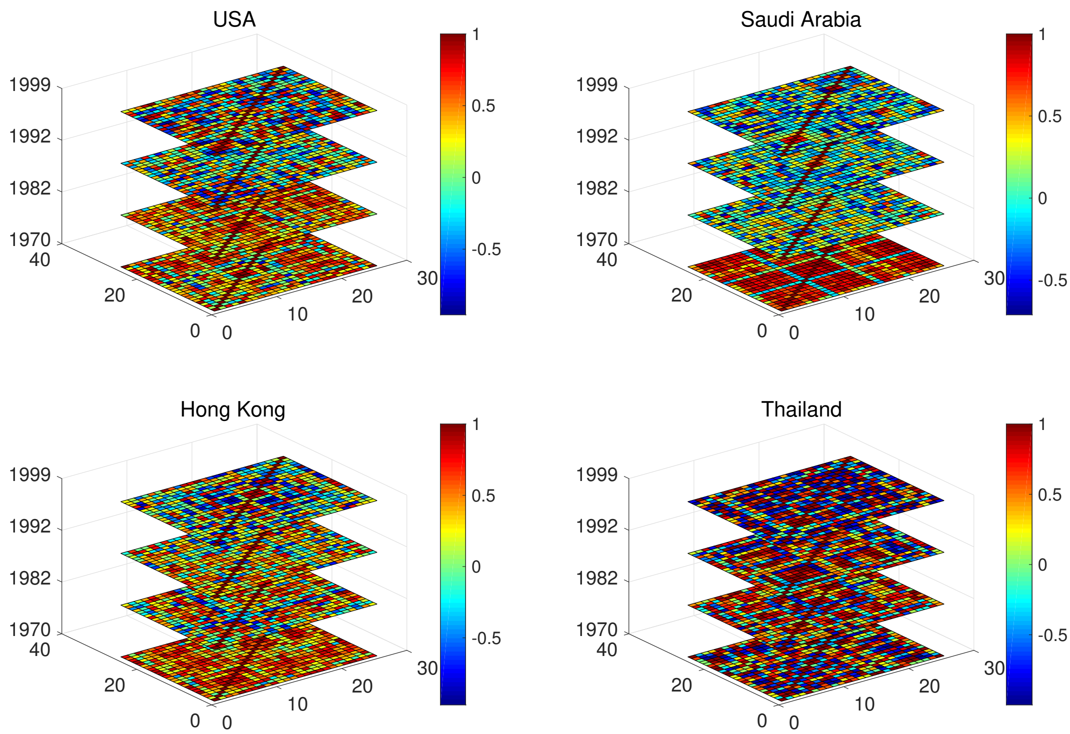


Fig. 4: International commodity trade correlation matrices for 1970 (first slice from bottom), 1982 (second slice from bottom), 1992 (third slice from bottom) and 1999 (top slice) for four contries.

S5. Description of Movies

The following is a list of the movies that have been included as supplementary materials and a brief discussion of their content.

Filename	Content Description
networks.mov	Object FPCs obtained as Fréchet integrals (11) for one randomly chosen simulation run for $n = 50$, using the model in (17) for time-varying networks in section 5.2.
mean_males.mov	Estimated Fréchet mean function for males represented as density functions indexed by year, derived from the yearly sample average of the quantile functions of the countries included in the mortality data in section 6.1 for each calendar year.
mean_females.mov	Estimated Fréchet mean function for females. The description is the same as for males.
mean_NY.mov	Estimated Fréchet mean function represented as time varying network adjacency matrices, obtained for each 20 minute time interval by averaging the network adjacency matrices over 363 daily networks for the New York taxi data as described in section 6.2.
week.mov	Fréchet integrals represented as network adjacency matrices for the dominant eigenfunctions, obtained from the analysis of the New York taxi data as described in section S3 of the supplement, for weekdays.
friday.mov	Same as previous, for Fridays.
Hol.mov	Same as previous, for weekends and special holidays.
trade.mov	Fréchet integrals represented as covariance matrices for the dominant eigenfunctions for the trade dataset as described in section 6.3.

S6. Data Descriptions

S6.1 Zones in Manhattan, New York

New York City Taxi and Limousine Commission (NYC TLC) provides records on pick-up and drop-off dates/times, pick-up and drop-off latitudes and longitudes, trip distances, itemized fares, rate types, payment types, and driver-reported passenger counts for yellow and green taxis. The data are available at http://www.nyc.gov/html/tlc/html/about/trip_record_data.shtml. The polygon shape files available at this website represent the boundaries zones for taxi pickups as delimited by the New York City Taxi and Limousine Commission (TLC). The latitudes and longitudes in New York are split into 6 boroughs: Bronx, Brooklyn, Newark Liberty International Airport, Manhattan, Queens and Staten Island. Since yellow taxis operate predominantly in Manhattan, we consider so-called towns in Manhattan which are further grouped into 10 zones as described in the following Table. We excluded the islands from our study. For a description of towns, we refer to Figure 5.

Zone	Towns
1	Inwood, Fort George, Washington Heights, Hamilton Heights, Harlem, East Harlem
2	Upper West Side, Morningside Heights, Central Park
3	Yorkville, Lenox Hill, Upper East Side
4	Lincoln Square, Clinton, Chelsea, Hell's Kitchen
5	Garment District, Theater District
6	Midtown
7	Midtown South
8	Turtle Bay, Murray Hill, Kips Bay, Gramecy Park, Sutton, Tudor, Medical City, Stuy Town
9	Meat packing district, Greenwich Village, West Village, Soho, Little Italy, China-Town, Civic center, Noho
10	Lower East Side, East Village, ABC Park, Bowery, Two Bridges, Southern tip, White Hall, Tribecca, Wall Street

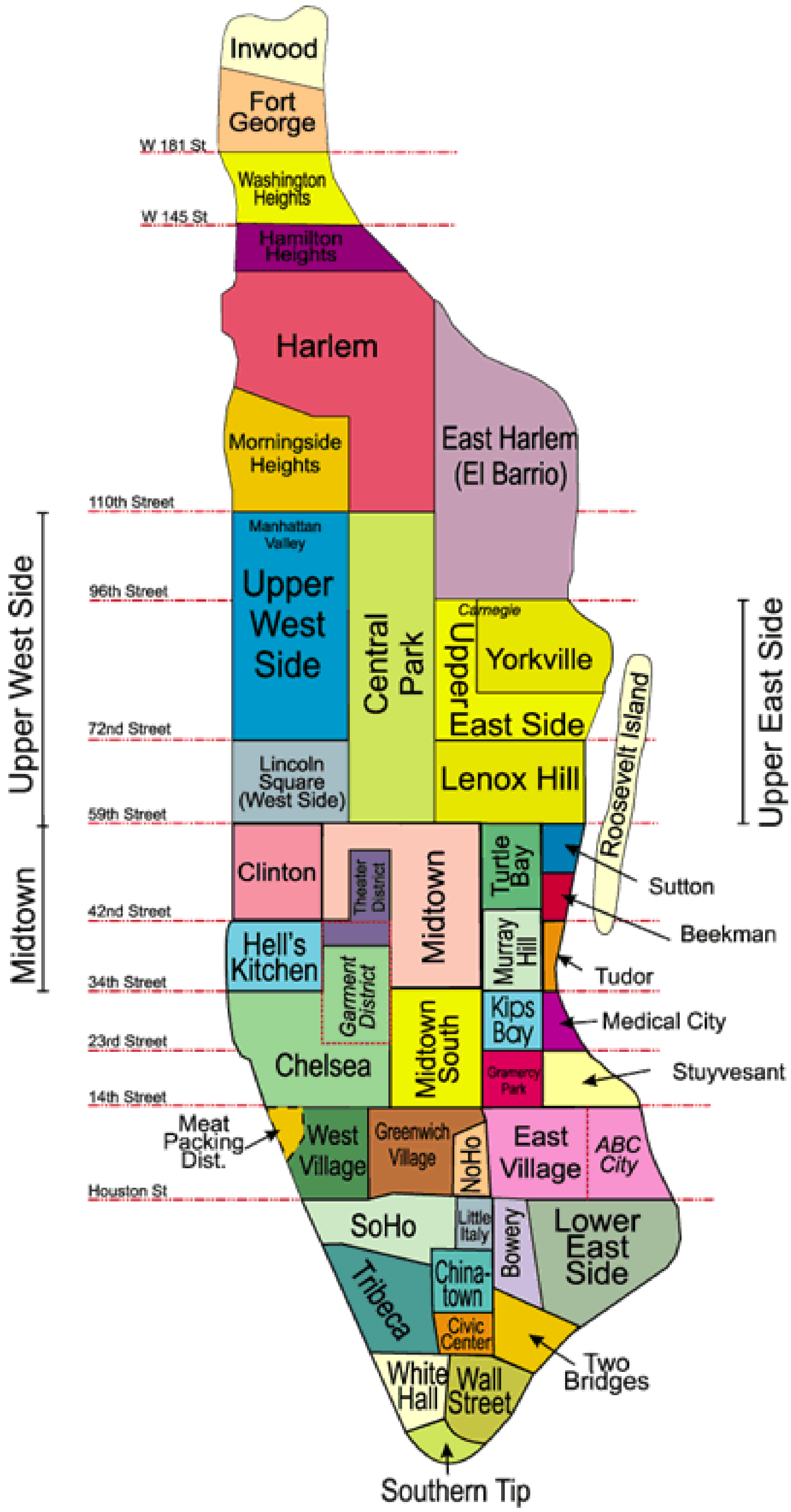


Fig. 5: Towns in Manhattan, New York.

S6.2 Trade Data

The countries chosen for the analysis were Morocco, Tunisia, Egypt, Canada, USA, Argentina, Brazil, Chile, Mexico, Venezuela, Dominican Republic, Israel, Japan, Cyprus, Lebanon, Saudi Arabia, United Arab Emirates, Turkey, Hong Kong, Indonesia, Korea Republic, Malaysia, Philippines, Singapore, Thailand, Taiwan, China, Belgium-Luxemburg, Denmark, France, Greece, Ireland, Italy, Netherlands, Portugal, Spain, UK, Austria, Finland, Norway, Sweden, Switzerland, Malta, Bulgaria, Australia and New Zealand. A list of the traded commodities can be found in the following table,

Number	Products
1	Sugar, Honey
2	Road Vehicles
3	Fruits and Vegetables
4	Non metallic Minerals Manufactures
5	Coffee, Tea, Cocoa, Spices
6	Tobacco and Tobacco Manufactures
7	Textiles, Yarns, Fabrics
8	Printed Books, Maps, Charts, Paper, Stationery
9	Beverages
10	Chemical Materials, Products
11	Machineries
12	Transport Equipments
13	Rubber
14	General Industrial Machinery
15	Dairy Products, Eggs
16	Fish and Seafood
17	Cereals
18	Petroleum
19	Dye
20	Medicines
21	Oil, Perfumes, Toilet, Cleansing
22	Paper, Paper Board, Articles
23	Iron and Steel
24	Manufacture of Metals
25	Power Generating Machinery
26	Telecommunications, Sound Recording, Reproducing Equipments