CENTRAL LIMIT THEOREM AND BOOTSTRAP APPROXIMATION IN HIGH DIMENSIONS WITH NEAR $1/\sqrt{n}$ RATES

BY MILES E. LOPES$^{1,*}$

$^1$University of California, Davis, melopes@ucdavis.edu

Non-asymptotic bounds for Gaussian and bootstrap approximation have recently attracted significant interest in high-dimensional statistics. This paper studies Berry-Esseen bounds for such approximations (with respect to the multivariate Kolmogorov distance), in the context of a sum of $n$ random vectors that are $p$-dimensional and i.i.d. Up to now, a growing line of work has established bounds with mild logarithmic dependence on $p$. However, the problem of developing corresponding bounds with near $n^{-1/2}$ dependence on $n$ has remained largely unresolved. Within the setting of random vectors that have sub-Gaussian entries, this paper establishes bounds with near $n^{-1/2}$ dependence, for both Gaussian and bootstrap approximation. In addition, the proofs are considerably distinct from other recent approaches.

1. Introduction. In recent years, the analysis of Berry-Esseen bounds for Gaussian and bootstrap approximation has become a quickly growing topic in high-dimensional statistics. Indeed, much of the work in this direction has been propelled by the fact that such approximations are essential tools for a wide variety inference problems. A survey of related applications and results may be found in (Belloni et al., 2018).

To briefly review the modern literature on multivariate Berry–Esseen bounds, a natural starting point is the seminal paper (Bentkus, 2003). In that work, Bentkus studied Gaussian approximation of a sum $S_n = n^{-1/2} \sum_{i=1}^n X_i$ of i.i.d. centered isotropic random vectors in $\mathbb{R}^p$. Letting $Y$ denote a centered Gaussian vector with $\mathbb{E}[YY^\top] = \mathbb{E}[X_1X_1^\top]$, and letting $\mathcal{A}$ denote the class of all Borel convex subsets of $\mathbb{R}^p$, Bentkus’ work showed that under suitable moment conditions, the measure of distance $\sup_{A \in \mathcal{A}} |\mathbb{P}(S_n \in A) - \mathbb{P}(Y \in A)|$ is at most of order $p^{7/4}n^{-1/2}$. (See also (Bentkus, 2005; Raič, 2019) for refinements and further references.) However, despite the strength of this result, it typically does not lend itself to applications where $p$ is large.

In high-dimensional settings, the paper (Chernozhukov, Chetverikov and Kato, 2013) achieved a breakthrough by demonstrating that if $\mathcal{A}$ is taken instead to be a certain class of hyperrectangles, then the corresponding measure of distance can be bounded at a rate that has a logarithmic dependence on $p$, such as $\log(p)n^{-1/8}$ (and similarly for bootstrap approximation). Subsequently, the papers (Chernozhukov, Chetverikov and Kato, 2017a) and (Chernozhukov et al., 2019), showed that when $\mathcal{A}$ includes all hyperrectangles, the rates for Gaussian and bootstrap approximation can be improved to $\log(p)n^{-1/6}$ and $\log(p)n^{-1/4}$ respectively. Meanwhile, a parallel series of works (Deng and Zhang (2020+); Kuchibhotla, Mukherjee and Banerjee (2018); Koike (2019); Deng (2020); Das and Lahiri (2020)) developed further improvements, by showing that Gaussian and bootstrap approximation can succeed asymptotically when $\log(p)^{\kappa} = o(n)$ and $2 \leq \kappa \leq 5$.

With regard to rates of Gaussian approximation that go beyond the $n^{-1/4}$ dependence on $n$, some results have appeared in (Lopes, Lin and Müller, 2020) and (Fang and Koike, 2020).

$^*$Supported in part by NSF grant DMS-1915786.

MSC2020 subject classifications: Primary 60F05, 62E17.

Keywords and phrases: central limit theorem, bootstrap, Berry-Esseen theorem, high dimensions.
The first of these papers considered a setting of “weak variance decay”, where $\text{var}(X_{ij}) = O(j^{-a})$ for all $1 \leq j \leq p$, with $a > 0$ being an arbitrarily small parameter. Under this type of structure, the authors established the rate $n^{-1/2+\delta}$ for arbitrarily small $\delta > 0$, when $\mathcal{A}$ is a certain class of hyperrectangles. In a different direction, the paper (Fang and Koike, 2020) dealt with a setting where $\mathcal{A}$ includes all hyperrectangles, and where the vector $X_1$ is isotropic. Within this setting, the authors established the rate $\log(p)^{3/2} \log(n)n^{-1/2}$ when $X_1$ has a continuous log-concave density, as well as the rate $\log(pn)^{4/3}n^{-1/3}$ when $X_1$ has sub-Gaussian entries (but need not have a density). In the current work, we focus on the latter case, and the main contribution of our first result (Theorem 2.1) is to establish a rate with near $n^{-1/2}$ dependence on $n$.

In addition to the work on Gaussian approximation described above, there are a few special cases where near $n^{-1/2}$ rates are known to be achievable via bootstrap approximation. First, in the setting of weak variance decay, it was shown in (Lopes, Lin and Müller, 2020) that the mentioned $n^{-1/2+\delta}$ rate holds for bootstrap approximation as well. Second, the paper (Chernozhukov et al., 2019) showed that near $n^{-1/2}$ rates can be achieved when bootstrap methods are used in particular ways. Namely, this was demonstrated in the case when the data have a symmetric distribution and Rademacher weights are chosen for the multiplier bootstrap, or when bootstrap quantiles are adjusted in a conservative manner. (See also (Deng, 2020) for further work in this direction.) In relation to these results, the current paper makes a second contribution in Theorem 2.3 by showing that a near $n^{-1/2}$ rate of bootstrap approximation holds, without relying variance decay, symmetry, or conservative adjustments.

Concerning the proofs, perhaps the most important point to discuss is the use of smoothing techniques. As is well known, these techniques are based on using a smooth function, say $\psi : \mathbb{R}^p \to \mathbb{R}$, depending on a set $A \subset \mathbb{R}^p$, such that $\mathbb{E}[\psi(S_n)] \approx \mathbb{P}(S_n \in A)$. Although these techniques are of fundamental importance, one of their drawbacks is that they often incur an extra smoothing error $|\mathbb{P}(S_n \in A) - \mathbb{E}[\psi(S_n)]|$, which must be balanced with errors from various other approximations. Moreover, this balancing process often turns out to be a bottleneck for the overall rate of distributional approximation.

As a way of avoiding this bottleneck, we use a smoothing function that arises “implicitly” as part of the Lindeberg interpolation scheme — which has the benefit that it does not create any smoothing error. More concretely, if $X_1, \ldots, X_n$ are non-Gaussian vectors and if $Y_1, \ldots, Y_n$ are Gaussian vectors, then this notion of smoothing is based on the fact that the probability $\mathbb{P}(\sum_{i=1}^k X_i + \sum_{j=k+1}^n Y_j \in A)$ can be equivalently written as $\mathbb{E}[\psi(\sum_{i=1}^k X_i)]$, for a particular smooth random function $\tilde{\psi}$ defined in terms of $Y_{k+1}, \ldots, Y_n$. (This is explained in detail in Section 4.1.) Furthermore, it turns out that the derivatives of $\tilde{\psi}$ may be controlled effectively, as a consequence of the work of (Bentkus, 1990). However, by itself, this type of smoothing does not seem to provide a way to handle every step of the Lindeberg interpolation, because the smoothing effect from the Gaussian vectors $Y_{k+1}, \ldots, Y_n$ runs out of steam when $k$ becomes close to $n$. To overcome this issue, a second important ingredient in the proof is the use of induction, which makes it possible to re-use good approximations from small values of $k$ at larger values of $k$. In particular, the use of induction here is influenced by the paper (Bentkus, 2003) (even though the approach to smoothing in that work is different).

**Notation.** We write $s \preceq r$ for two vectors $s, r \in \mathbb{R}^p$ satisfying the inequalities $s_j \leq r_j$ for all $1 \leq j \leq p$. A scalar random variable $U$ is said to be sub-Gaussian if it has a finite $\psi_2$-Orlicz norm, defined by $\|U\|_{\psi_2} = \inf\{t > 0 | \mathbb{E} [\exp(t^2/2)] \leq 2\}$. If $V$ is another random variable that is equal in distribution to $U$, then we write $V \equiv U$. If $x$ is a vector, matrix, or tensor with real entries, we use $\|x\|_{\infty}$ to refer to the maximum absolute value of the entries, and $\|x\|_1$ to refer to the sum of the absolute values of the entries. Also, the identity matrix in
\( \mathbb{R}^{p \times p} \) is denoted by \( I_p \). Throughout the paper, the symbol \( c \) will denote a positive absolute constant whose value may vary at each occurrence. (Different symbols will be used when it is necessary to track constants.) Lastly, in order to simplify presentation, we will use the function \( \log(t) = \max\{\log(t), 1\} \), where \( \log \) is the ordinary natural logarithm.

2. Main results. The following theorem is the core result of the paper. Later on, a Gaussian comparison result (Theorem 2.2) and a bootstrap approximation result (Theorem 2.3) are obtained as extensions. The main aspects of the proof of Theorem 2.1 are given in Section 3.

**Theorem 2.1 (Gaussian approximation).** There is an absolute constant \( \bar{C} > 0 \), such that the following holds for all \( n \) and \( p \): Let \( X_1, \ldots, X_n \in \mathbb{R}^p \) be centered i.i.d. random vectors, and suppose that \( \nu = \max_{1 \leq j \leq p} \|X_{1j}/\sqrt{\text{var}(X_{1j})}\|_{\psi_2} \) is finite. In addition, let \( \rho \) be the smallest eigenvalue of the correlation matrix of \( X_1 \), and suppose that \( \rho > 0 \). Lastly, let \( Y \in \mathbb{R}^p \) be a centered Gaussian random vector with \( \mathbb{E}[YY^\top] = \mathbb{E}[X_1X_1^\top] \). Then,

\[
\sup_{r \in \mathbb{R}^p} \left| \mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \leq r \right) - \mathbb{P} (Y \leq r) \right| \leq C \left( \frac{\rho^{p/2}}{n^{p/2}} \right) \log(pn)^4 \log(n) \frac{1}{n^{1/2}}.
\]

Remarks. To discuss some of the characteristics of the bound, it should be mentioned that the reliance on the sub-Gaussian entries of \( X_1 \) can be relaxed. In particular, a corresponding result for sub-exponential entries can be obtained with a larger power of \( \log(pn) \).

Next, there are a few items to consider with regard to the dependence on the parameter \( \rho \). First, it is important to clarify that \( \rho \) is often much larger than the smallest eigenvalue of the covariance matrix \( \mathbb{E}[X_1X_1^\top] \), say \( \lambda \). This is easiest to see in the context of uncorrelated variables, where \( \rho = 1 \) holds for arbitrarily small \( \lambda > 0 \). More generally, in the context of strongly correlated variables, the parameter \( \rho \) need not be very small either. For instance, in the case when \( \text{cor}(X_{1i}, X_{1j}) = 0.9 \) for all \( i \neq j \), it follows that \( \rho = 0.1 \) (regardless of the dimension \( p \)). With regard to the Gaussian approximation results in (Fang and Koike, 2020), it is difficult to make a comparison with respect to the dependence on \( \rho \), since those results are formulated in an isotropic case where \( \rho = 1 \). Meanwhile, it should be noted that the Gaussian approximation result in (Chernozhukov et al., 2019) does not rely on the condition \( \rho > 0 \). Instead, that result depends on how well separated the parameter \( \zeta^2 = \min_{1 \leq j \leq p} \text{var}(X_{1j}) \) is from 0. These different relative merits also apply to the Gaussian comparison and bootstrap approximation results in that work, vis-a-vis Theorems 2.2 and 2.3 given below.

At an informal level, if \( \rho \) is well separated from 0, this can be interpreted to mean that the distribution of \( X_1 \) is “fully high-dimensional” — which is precisely the case we are interested in. On the other hand, if the correlation matrix of \( X_1 \) has some eigenvalues that are very small, this is an indication that the distribution of \( X_1 \) has some low-dimensional structure, and in that case, it may be preferable to pursue a different approach that takes the structure directly into account, such as in (Lopes, Lin and Müller, 2020). Nevertheless, it turns out that it is possible to extend Theorem 2.1 to handle the case when \( \rho = 0 \), provided that the correlation matrix of \( X_1 \) is close to a positive definite matrix in an entrywise sense. However, this will not be needed in order to develop the bootstrap approximation result later on.

2.1. Gaussian comparison. Our second result provides a bound on the Kolmogorov distance between two Gaussian vectors in terms of a normalized \( \ell_\infty \)-distance between their covariance matrices. In addition to being of basic interest by itself, this result will serve as a bridge to connect the Gaussian approximation result in Theorem 2.1 with the bootstrap approximation result in Theorem 2.3.
THEOREM 2.2 (Gaussian comparison). There is an absolute constant $c > 0$, such that the following holds for all $p$: Let $Y$ and $Z$ be centered Gaussian vectors in $\mathbb{R}^p$, having respective covariance matrices $\Sigma_Y$ and $\Sigma_Z$. In addition, let $p$ be the smallest eigenvalue of the correlation matrix of $Y$, and suppose that $p > 0$. Lastly, let $D = \text{diag}(\Sigma_{Y11}, \ldots, \Sigma_{yp})$, and let

$$\Delta = \|D^{-1/2}(\Sigma_Z - \Sigma_Y)D^{-1/2}\|_\infty.$$ 

Then,

$$\sup_{r \in \mathbb{R}^p} \left| \mathbb{P}(Z \leq r) - \mathbb{P}(Y \leq r) \right| \leq \left( \frac{c}{p} \right) \log(p) \log\left( \frac{1}{\rho} \right) \Delta.$$  

Remarks. Although the bound depends on the invertibility of $\Sigma_Y$, it is important to note that the bound does not depend on the invertibility of $\Sigma_Z$. This is a key property in the context of bootstrap approximation, where $\Sigma_Z$ will represent a sample covariance matrix that is possibly non-invertible. Another comment to make about Theorem 2.2 is its relation to Corollary 5.1 of the paper (Chernozhukov et al., 2019). That result establishes a Gaussian comparison bound of the form $c(\varsigma) \log(p) \sqrt{\|\Sigma_Y - \Sigma_Z\|_\infty}$, where the parameter $\varsigma^2 = \min_{1 \leq j \leq p} \Sigma_{jj}$ is assumed to be positive, and the constant $c(\varsigma)$ depends only on $\varsigma$. In this connection, the essential point to notice is that the bound (2) has a near-linear dependence on the parameter $\Delta$.

2.2. Bootstrap approximation. For a set of observations $X_1, \ldots, X_n \in \mathbb{R}^p$, the associated sample covariance matrix is defined as

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})^\top,$$

where $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. In terms of this matrix, the Gaussian multiplier bootstrap method developed by (Chernozhukov, Chetverikov and Kato, 2013) is based on generating a set of independent random vectors $X_1^*, \ldots, X_n^* \in \mathbb{R}^p$ from the Gaussian distribution $N(0, \hat{\Sigma})$. The general purpose of this method is to use the distribution of the sum $X_1^* + \cdots + X_n^*$ (conditional on the original observations) as an approximation to the distribution of the sum $X_1 + \cdots + X_n$. Accordingly, we will use the notation $\mathbb{P}(\cdot | X)$ to refer to probability that is conditional on $X_1, \ldots, X_n$.

THEOREM 2.3 (Bootstrap approximation). There is an absolute constant $c > 0$, such that the following holds for all $n$ and $p$: Suppose that the conditions of Theorem 2.1 hold, and let $X_1^*, \ldots, X_n^*$ be independent Gaussian random vectors drawn from $N(0, \hat{\Sigma})$. Then, the following event holds with probability at least $1 - \frac{1}{n}$,

$$\sup_{r \in \mathbb{R}^p} \left| \mathbb{P}\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i^* \leq r \bigg| X \right) - \mathbb{P}\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \leq r \right) \right| \leq \frac{c(\frac{p^{1/2}}{\rho^{1/2}}) \log(pn)^4 \log(n)}{n^{1/2}}.$$

Remarks. This result follows directly from Theorem 2.1 and Theorem 2.2 by letting $\hat{\Sigma}$ play the role of $\Sigma_Z$, and letting $\Sigma = \mathbb{E}[X_1X_1^\top]$ play the role of $\Sigma_Y$. To provide a bit more detail, we need only consider the case when $n \geq \log(pn)$, for otherwise there is nothing to prove. In this case, there is an absolute constant $c > 0$, such that the random variable $\hat{\Delta} = \|D^{-1/2}(\hat{\Sigma} - \Sigma)D^{-1/2}\|_\infty$ satisfies the bound $\hat{\Delta} \leq c\nu^2 \sqrt{\log(pn)n^{-1/2}}$ with probability at least $1 - \frac{1}{n}$, as recorded in Lemma 7.1 of Section 7.

Outline. After a high-level proof of Theorem 2.1 is given in Section 3, some preparatory items are developed in Section 4, which will be used in the more technical arguments given in Section 5. Later on, Theorem 2.2 is proven in Section 6, and various background results are summarized in Section 7.
3. The main steps in the proof of Theorem 2.1. Without loss of generality, we may assume that the covariance matrix $E[X_1 X_1^\top]$ has all ones along the diagonal, because the Kolmogorov distance is invariant to diagonal rescaling. Also, for future reference, it will be useful to take note of the general bounds $\rho \leq 1$ and $\nu \geq 1$, which are implied by the definitions of $\rho$ and $\nu$.

To lay out the beginning of the proof, let $Y_1, \ldots, Y_n \in \mathbb{R}^p$ be i.i.d. copies of $Y$ that are independent of $X_1, \ldots, X_n$. In order to write various partial sums, we use the following notation for $1 \leq k \leq k' \leq n$,

$$S_{k:k'}(X) = n^{-1/2}(X_k + \cdots + X_{k'})$$
$$S_{k:k'}(Y) = n^{-1/2}(Y_k + \cdots + Y_{k'}).$$

In addition, it will be convenient to denote the Kolmogorov distance between $S_{1:k}(X)$ and $S_{1:k}(Y)$ as

$$D_k = \sup_{r \in \mathbb{R}^p} \left| \Pr(S_{1:k}(X) \leq r) - \Pr(S_{1:k}(Y) \leq r) \right|.$$ 

To write down a basic form of Lindeberg interpolation for bounding $D_n$, define the following quantities for any $r \in \mathbb{R}^p$, 

$$\delta_k^X(r) = \Pr(S_{1:k-1}(X) + \frac{1}{\sqrt{n}}X_k + S_{k+1:n}(Y) \leq r) - \Pr(S_{1:k-1}(X) + S_{k+1:n}(Y) \leq r),$$
$$\delta_k^Y(r) = \Pr(S_{1:k-1}(X) + \frac{1}{\sqrt{n}}Y_k + S_{k+1:n}(Y) \leq r) - \Pr(S_{1:k-1}(X) + S_{k+1:n}(Y) \leq r).$$

This notation yields the interpolation

$$\Pr(S_{1:n}(X) \leq r) - \Pr(S_{1:n}(Y) \leq r) = \sum_{k=1}^n \delta_k^X(r) - \delta_k^Y(r).$$

Next, define the supremum of the $k$th difference

$$\delta_k = \sup_{r \in \mathbb{R}^p} |\delta_k^X(r) - \delta_k^Y(r)|,$$

which leads to the bound

$$D_n \leq \delta_1 + \cdots + \delta_n.$$

However, rather than working directly with the entire sum $\delta_1 + \cdots + \delta_n$, we will begin with a lemma that reduces the problem to bounding $\delta_1 + \cdots + \delta_{n-m}$ for an integer $m$ that will be carefully chosen later on.

**Lemma 3.1.** There is an absolute constant $c_1 > 0$ such that the following holds for all $n, p \geq 1$ and $1 \leq m \leq n$: If the conditions of Theorem 2.1 hold, then

$$D_n \leq c_1 \nu \sqrt{\frac{m}{n} \log(pn)} + 3(\delta_1 + \cdots + \delta_{n-m}).$$

It is worthwhile to proceed straight to the proof of this lemma, since the argument is fairly short, and since the notation in it will be used later on.

**Proof.** As a temporary shorthand, let $\zeta = S_{1:n}(X)$ and $\xi = S_{1:n-m}(X) + S_{n-m+1:n}(Y)$. Observe that the Kolmogorov distance between $\zeta$ and $S_{1:n}(Y)$ is at most $\delta_1 + \cdots + \delta_{n-m}$, which gives

$$D_n \leq \sup_{r \in \mathbb{R}^p} \left| \Pr(\zeta \leq r) - \Pr(\xi \leq r) \right| + (\delta_1 + \cdots + \delta_{n-m}).$$
To control the first term on the right side, define the corner set associated with a fixed \( r \in \mathbb{R}^p \) and \( t \in \mathbb{R} \),

\[
\mathcal{C}(r,t) = \left\{ x \in \mathbb{R}^p \left| x \leq r + t\mathbf{1} \right. \right\},
\]

where \( \mathbf{1} \in \mathbb{R}^p \) is the all-ones vector. Also, define an associated boundary set of “width” \( 2t \),

\[
\partial \mathcal{C}(r,t) = \mathcal{C}(r,t) \setminus \mathcal{C}(r,-t).
\]

In terms of this notation, we have the following basic inequality (Lemma 7.3), which holds for any \( r \in \mathbb{R}^p \) and any \( t > 0 \),

\[
|\mathbb{P}(\zeta \leq r) - \mathbb{P}(\zeta \leq r)| \leq \mathbb{P}(\zeta \in \partial \mathcal{C}(r,t)) + \mathbb{P}(\|\zeta - \xi\|_\infty \geq t).
\]

Next, we control the probability that \( \xi \) hits \( \partial \mathcal{C}(r,t) \) by essentially replacing \( \xi \) with \( S_{1:n}(Y) \). To do this, again note that the Kolmogorov distance between \( \xi \) and \( S_{1:n}(Y) \) is at most \( \delta_1 + \cdots + \delta_{n-m} \), and so

\[
\mathbb{P}(\zeta \in \mathcal{C}(r,t)) \leq \mathbb{P}(S_{1:n}(Y) \in \mathcal{C}(r,t)) + (\delta_1 + \cdots + \delta_{n-m}).
\]

Similarly

\[
\mathbb{P}(\zeta \in \mathcal{C}(r,-t)) \geq \mathbb{P}(S_{1:n}(Y) \in \mathcal{C}(r,-t)) - (\delta_1 + \cdots + \delta_{n-m}),
\]

and then combining gives

\[
\mathbb{P}(\zeta \in \partial \mathcal{C}(r,t)) \leq \mathbb{P}(S_{1:n}(Y) \in \partial \mathcal{C}(r,t)) + 2(\delta_1 + \cdots + \delta_{n-m}).
\]

In turn, Nazarov’s Gaussian anti-concentration inequality (Lemma 7.2) gives

\[
\mathbb{P}(S_{1:n}(Y) \in \partial \mathcal{C}(r,t)) \leq c t \sqrt{\log(p)},
\]

where we have made use of the reduction that \( \mathbb{E}[YY^\top] \) has all ones along the diagonal. Lastly, observe that by a sub-Gaussian tail bound (Lemma 7.1), if we take \( t = c \nu \sqrt{\frac{\nu \log(p \nu)}{n}} \) for a sufficiently large absolute constant \( c > 0 \), then the coupling probability \( \mathbb{P}(\|\zeta - \xi\|_\infty > t) \) is at most \( \frac{\nu}{n} \). Furthermore, given that the parameter \( \nu \) is at least 1, it follows that the quantity \( \frac{\nu}{n} \) is of negligible order in comparison to \( t \sqrt{\log(p)} \).

\( \square \)

**Comments on induction.** In order to prove Theorem 2.1, we will use a form of strong induction. Specifically, for a given absolute constant \( \bar{C} > 0 \), and given integers \( n \) and \( p \), the associated induction hypothesis is that the inequality \((H_k(\bar{C}))\) below holds simultaneously for all \( k = 1, \ldots, n-1 \),

\[
(H_k(\bar{C})) \quad D_k \leq \frac{\bar{C}(\nu^{k/2}) \log(pk)^4 \log(k)}{k^{1/2}}.
\]

Although it is common in high-dimensional statistics to think of \( n \) and \( p \) as growing together, it is worth clarifying that the inductive approach here is based on showing that, for any fixed \( p \), the entire sequence \( H_1(\bar{C}), H_2(\bar{C}), \ldots \) holds. Hence, because \( p \) is arbitrary, it will follow that the statement of the main result holds for all pairs \((n,p)\).
3.1. Proof of Theorem 2.1. Observe that if \( \tilde{C} \geq \sqrt{2} \), then it is clear that \( H_1(\tilde{C}) \) and \( H_2(\tilde{C}) \) hold. To carry out the induction, fix any \( n \geq 3 \), and suppose that \( H_1(\tilde{C}), \ldots, H_{n-1}(\tilde{C}) \) hold for some absolute constant \( \tilde{C} \geq \sqrt{2} \). Our goal is now to show that \( H_n(\tilde{C}) \) holds (with the same value of \( \tilde{C} \)). The main tool for this purpose is the proposition below, whose proof is deferred to Section 5.

**Proposition 3.2.** There is a positive absolute constant \( c_2 \) such that the following holds for all \( n \geq 3 \), \( p \geq 1 \), and \( 1 \leq m \leq n/3 \): If the the conditions of Theorem 2.1 hold, and if \( H_1(\tilde{C}), \ldots, H_{n-1}(\tilde{C}) \) hold for some absolute constant \( \tilde{C} \geq \sqrt{2} \), then

\[
\delta_1 + \cdots + \delta_{n-m} \leq \frac{c_2 \left( \frac{\nu^{3/2}}{p} \right) \log(pm)^4 \log(n)}{n^{1/2}} + \frac{c_2 \tilde{C} \left( \frac{\nu}{p^3} \right) \log(pm)^7 \log(n)}{n^{1/2} m^{1/2}}.
\]

At a high level, Lemma 3.1 and Proposition 3.2 reduce the proof of Theorem 2.1 to exhibiting suitable values of \( m \) and \( \tilde{C} \). To proceed, let \( c_1 \) and \( c_2 \) be the absolute constants in the statements of these results. We may assume without loss of generality that \( c_2 = c_1 \) and \( c_1 \geq 1 \), because these results remain true if \( c_1 \) and \( c_2 \) are both replaced by \( \max\{c_1, c_2, 1\} \). Next, let \( 1 \leq m \leq n/3 \), and define the quantities \( \alpha, \beta, \) and \( \gamma \) according to

\[
\alpha = \frac{c_1 \nu m^{1/2} \log(pm)}{n^{1/2}},
\]

\[
\beta = \frac{c_1 \left( \frac{\nu^{3/2}}{p} \right) \log(pm)^4 \log(n)}{n^{1/2}} + \frac{c_1 \left( \frac{\nu}{p^3} \right) \log(pm)^7 \log(n)}{n^{1/2} m^{1/2}},
\]

\[
\gamma = \frac{\tilde{C} \left( \frac{\nu^{3/2}}{p^{3/2}} \right) \log(pm)^4 \log(n)}{n^{1/2}}.
\]

In terms of this notation, Lemma 3.1 and Proposition 3.2 give the bound

\[
D_n \leq \alpha + 3\beta.
\]

Therefore, in order to show \( H_n(\tilde{C}) \), it is enough to show that there exist choices of \( m \) and \( \tilde{C} \) for which

\[
\alpha + 3\beta \leq \gamma.
\]

(11) \( \alpha \leq \frac{2c_1 \left( \frac{\nu^{3/2}}{p^{3/2}} \right) \log(pm)^4}{n^{1/2}} \)

(12) \( 3\beta \leq \frac{3c_1 \left( \frac{\nu^{3/2}}{p} \right) \log(pm)^4 \log(n)}{n^{1/2}} + \frac{\left( \frac{3c_1 \tilde{C}}{\kappa} \right) \left( \frac{\nu^{3/2}}{p^{3/2}} \right) \log(pm)^4 \log(n)}{n^{1/2}}. \)
(Note that in (11), the prefactor of 2 is introduced so that the ceiling function in (10) can be ignored.) Since \( \nu \geq 1 \) and \( \rho \leq 1 \), we have \( \frac{\nu^{3/2}}{\rho^{3/2}} \leq \frac{2c_1}{\rho^{3/2}} \), which implies
\[
\alpha + 3\beta \leq \left(2\kappa c_1 + 3c_1 + \frac{3c_1 C}{\kappa}\right) \cdot \frac{\gamma}{C}.
\]
Thus, in order to show \( \alpha + 3\beta \leq \gamma \), it suffices to select \( \kappa \) and \( \bar{C} \) in terms of \( c_1 \) so that
\[
2\kappa c_1 + 3c_1 + \frac{3c_1 C}{\kappa} \leq \bar{C}.
\]
Likewise, if we put
\[
\kappa = \sqrt{\frac{2}{3} \bar{C}},
\]
then \( \bar{C} \) should be chosen to satisfy
\[
(2\sqrt{6c_1}) \sqrt{\bar{C}} + 3c_1 \leq \bar{C}.
\]
This is a quadratic inequality in \( \sqrt{\bar{C}} \), which holds when
\[
\bar{C} \geq \left(\sqrt{6c_1} + \sqrt{6c_1^2 + 3c_1}\right)^2.
\]
In particular, this is compatible with the condition \( \bar{C} \geq \sqrt{2} \) mentioned earlier, since \( c_1 \geq 1 \). Moreover, since the right side of (14) is purely a function of \( c_1 \), the only remaining consideration is to make sure that (14) allows for a feasible choice of \( m \leq n/3 \). (Note that \( m \) is now determined by \( \bar{C} \) through (10) and (13).) To do this, we may assume without loss of generality that the inequality
\[
n^{1/2} \geq \bar{C} \left(\frac{\rho^{3/2}}{\rho^{3/2}}\right) \log(pm)^4 \log(n)
\]
holds, for otherwise \( H_n(\bar{C}) \) is trivially true. Comparing this inequality with (10) shows that the condition \( m \leq n/3 \) holds, for instance, when \( 2\sqrt{(3/2)\bar{C}} \leq \bar{C}/\sqrt{3} \), i.e. when \( \bar{C} \geq 18 \). But at the same time, the right side of (14) is already greater than 18, and so it suffices to take \( \bar{C} \) equal to the right side of (14).

4. Preparatory items. The section develops the notation and key objects that will be needed to prove Proposition 3.2 in Section 5.

4.1. Implicit smoothing. The main idea in this subsection is represent the quantities \( \delta^X_k(r) \) and \( \delta^Y_k(r) \) in terms of a certain implicit Gaussian smoothing function. We use the word “implicit”, because the smoothing function is automatically built into the Lindeberg interpolation through the Gaussian partial sums.

To proceed, let \( \zeta \sim N(0, I_p) \) be a standard Gaussian vector in \( \mathbb{R}^p \), and for any fixed \( r, s \in \mathbb{R}^p \) and \( \epsilon > 0 \), define
\[
\varphi_\epsilon(s, r) = \mathbb{P}(s + \epsilon \zeta \leq r) = \prod_{j=1}^p \Phi \left( \frac{r_j - s_j}{\epsilon} \right).
\]
When \( r \) is held fixed, the function \( \varphi_\epsilon(\cdot, r) \) is a smoothed version of the indicator \( s \mapsto 1\{s \leq r\} \), with \( \epsilon \) playing the role of a smoothing parameter. Next, for each \( k = 1, \ldots, n - 1 \), define
\[
\epsilon_k = \sqrt{\frac{n-k}{n}} \rho.
\]
The parameter $\epsilon_k$ is used in order to simplify the following (distributional) decomposition of the Gaussian vector $S_{k+1:n}(Y)$,

$$S_{k+1:n}(Y) \overset{d}{=} \epsilon_k V_{k+1} + \sqrt{\frac{n-k}{n}} W_{k+1},$$

where $V_{k+1} \sim N(0, I_p)$ and $W_{k+1} \sim N(0, R - \rho I_p)$ are independent, and $R$ is the correlation matrix of $X_1$. (Here, we continue to work under the reduction that $E[X_1X_1^\top] = R$. Also, note that the vectors $V_{k+1}$ and $W_{k+1}$ may be taken to be independent of $X_1, \ldots, X_n$.) Consequently, if we let

$$\hat{r}_{k+1} = r - \sqrt{\frac{n-k}{n}} W_{k+1},$$

then we can connect $\varphi_{\epsilon_k}$ to the partial sums in the Lindeberg interpolation through the following exact relation

$$P\left(S_{1:k}(X) + S_{k+1:n}(Y) \leq r\right) = E\left[\varphi_{\epsilon_k}(S_{1:k}(X), \hat{r}_{k+1})\right].$$

In turn, this relation allows us to express $\delta_k^X(r)$ in terms of $\varphi_{\epsilon_k}$ for $k = 1, \ldots, n - 1$,

$$\delta_k^X(r) = E\left[\varphi_{\epsilon_k}\left(S_{1:k-1}(X) + \frac{1}{\sqrt{n}} X_k, \hat{r}_{k+1}\right) - \varphi_{\epsilon_k}\left(S_{1:k-1}(X), \hat{r}_{k+1}\right)\right].$$

The formula (18) is the key item to take away from the current subsection. The corresponding expression for $\delta_k^Y(r)$ is nearly identical, with the only change being that the single occurrence of $X_k$ in (18) is replaced with $Y_k$.

4.2. Moment matching. By expanding the function $\varphi_{\epsilon_k}(\cdot, \hat{r}_{k+1})$ to second order at the point $S_{1:k-1}(X)$, we have the moment-matching formulas

$$\delta_k^X(r) = E[L_k^X(r)] + E[Q_k^X(r)] + E[R_k^X(r)]$$

(19)

where $\delta_k^X(r)$ are defined as follows. Specifically, if all derivatives are understood as being with respect to the first argument of $\varphi_{\epsilon_k}$, then

$$L_k^X(r) = \left\langle \nabla \varphi_{\epsilon_k}(S_{1:k-1}(X), \hat{r}_{k+1}), n^{-1/2} X_k \right\rangle$$

(20)

$$Q_k^X(r) = \frac{1}{2} \left\langle \nabla^2 \varphi_{\epsilon_k}(S_{1:k-1}(X), \hat{r}_{k+1}), n^{-1} X_k X_k^\top \right\rangle$$

(21)

$$R_k^X(r) = \frac{(1-r)^2}{2} \left\langle \nabla^3 \varphi_{\epsilon_k}(S_{1:k-1}(X) + \frac{r}{\sqrt{n}} X_k, \hat{r}_{k+1}), n^{-3/2} X_k^\otimes 3 \right\rangle,$$

(22)

with $r$ being a Uniform$[0,1]$ random variable that is independent of all other random variables. The notation $\nabla^3 \varphi_{\epsilon_k}(s, r)$ refers to the tensor in $\mathbb{R}^{p \times p \times p}$ whose entries are comprised by all possible three-fold partial derivatives of $\varphi_{\epsilon_k}(\cdot, r)$ at the point $s$. Also, we use $\langle \cdot \rangle$ to denote the entrywise inner product on vectors, matrices, and tensors. Lastly, the terms $L_k^Y(r)$, $Q_k^Y(r)$ and $R_k^Y(r)$ associated with $\delta_k^Y(r)$ in (20) only differ from those above insofar as each appearance of $X_k$ on the right sides of (21), (22), and (23) is replaced by $Y_k$.

The classical idea of the Lindeberg interpolation is that if (20) is subtracted from (19), then the first and second order terms cancel, because $X_k$ and $Y_k$ have matching mean vectors and covariance matrices. This leads to the relation

$$\delta_k^X(r) - \delta_k^Y(r) = E[R_k^X(r)] - E[R_k^Y(r)].$$

(23)

Hence, in order to control the supremum $\delta_k = \sup_{r \in \mathbb{R}^p} |\delta_k^X(r) - \delta_k^Y(r)|$ in (5), it remains to bound the expected remainders uniformly with respect to $r \in \mathbb{R}^p$, and this is handled in the next section.
5. Bounds for $\delta_k$, and the proof of Proposition 3.2. The next lemma handles $\delta_k$ for $k = 2, \ldots, n - 1$. This lemma is of special significance to the overall structure of the proof of Theorem 2.1, because it sets up the opportunity to apply the induction hypothesis to $D_{k-1}$. Apart from this, the quantity $\delta_1$ will be handled separately in Lemma 5.2 later on. (It will not be necessary to handle $\delta_n$, due to Lemma 3.1.) At the end of the section, the proof of Proposition 3.2 will be given.

**Lemma 5.1.** There is an absolute constant $c > 0$ such that the following holds for all $n \geq 3$, $p \geq 1$, and $2 \leq k \leq n - 1$: If the conditions of Theorem 2.1 hold, then

$$\delta_k \leq \frac{c^{1/2\log(pn)^3}}{\epsilon_k^n n^{1/4}} \left(\epsilon_k \log(pn) \sqrt{n} + D_{k-1} + \frac{1}{pn}\right).$$

**Proof.** From the previous section, we have the following bound on $\delta_k$,

$$\delta_k \leq \sup_{r \in \mathbb{R}^p} \mathbb{E}\left[\|R_k^X(r)\|\right] + \sup_{r \in \mathbb{R}^p} \mathbb{E}\left[\|R_k^Y(r)\|\right].$$

The current proof will only establish a bound on $\sup_{r \in \mathbb{R}^p} \mathbb{E}\left[\|R_k^X(r)\|\right]$, since the argument is the same for $\sup_{r \in \mathbb{R}^p} \mathbb{E}\left[\|R_k^Y(r)\|\right]$. To begin, define the random vector $\tilde{r}_{k+1} = \tilde{r}_{k+1} - \frac{r}{\sqrt{n}} X_k$, and for any fixed $\varepsilon > 0$, define the event

$$A_k(\varepsilon) = \left\{S_{1:k-1}(X) \in \partial C(\tilde{r}_{k+1}, \varepsilon)\right\}.$$

Below, we will separately analyze $R_k^X(r)$ on the event $A_k(\varepsilon)$ and its complement $A_k^c(\varepsilon)$, via

$$\mathbb{E}\left[\|R_k^X(r)\|\right] = \mathbb{E}\left[\|R_k^X(r)\|1\{A_k(\varepsilon)\}\right] + \mathbb{E}\left[\|R_k^X(r)\|1\{A_k^c(\varepsilon)\}\right].$$

**Handling the remainder on $A_k(\varepsilon)$.** By applying Hölder’s inequality to the definition of $R_k^X(r)$ in (23), we have

$$|R_k^X(r)|1\{A_k(\varepsilon)\} \leq \frac{1}{n^{3/2}} \left(\sup_{s, r \in \mathbb{R}^p} \|\nabla^3 \varphi_{\varepsilon_k}(s, r)\|_1\right) \cdot \|X_k\|_\infty^3 \cdot 1\{A_k(\varepsilon)\},$$

where $\|\nabla^3 \varphi_{\varepsilon_k}(s, r)\|_1$ refers to the sum of the absolute values of the entries in the 3-tensor $\nabla^3 \varphi_{\varepsilon_k}(s, r)$. Crucially, it is known from (Bentkus, 1990, Theorem 3) that

$$\sup_{s, r \in \mathbb{R}^p} \|\nabla^3 \varphi_{\varepsilon_k}(s, r)\|_1 \leq \frac{c \log(p)^{3/2}}{\epsilon_k^n}.$$

To be precise, the result (Bentkus, 1990, Theorem 3) is stated for functions that are slightly different from $\varphi_{\varepsilon_k}(r, s)$, but a more recent statement of the result that matches the form of (28) can be found in (O’Donnell, Servedio and Tan, 2019, Theorem 6.5).

Thus, it remains to control the expectation $\mathbb{E}\left[\|X_k\|_\infty^3 1\{A_k(\varepsilon)\}\right]$. Noting that $S_{1:k-1}(X)$ is independent of $\tilde{r}_{k+1}$ and $X_k$, we have

$$\mathbb{E}\left[\|X_k\|_\infty^3 1\{A_k(\varepsilon)\}\right] = \mathbb{E}\left[\|X_k\|_\infty^3 \mathbb{P}\left(S_{1:k-1}(X) \in \partial C(\tilde{r}_{k+1}, \varepsilon) \mid \tilde{r}_{k+1}, X_k\right)\right]$$

$$\leq \mathbb{E}\left[\|X_k\|_\infty^3 \left(\mathbb{P}\left(S_{1:k-1}(Y) \in \partial C(\tilde{r}_{k+1}, \varepsilon) \mid \tilde{r}_{k+1}, X_k\right) + 2D_{k-1}\right)\right]$$

$$\leq \mathbb{E}\left[\|X_k\|_\infty^3 \left(c \varepsilon \sqrt{\frac{n}{k-1}} \sqrt{\log(p)} + 2D_{k-1}\right)\right],$$

where we note that $S_{1:k-1}(X)$ has been replaced with $S_{1:k-1}(Y)$ at the price of $2D_{k-1}$, and Nazarov’s Gaussian anti-concentration inequality (Lemma 7.2) has been used in the
last step. Combining the last several steps with the bound \( \mathbb{E}[\|X_k\|_\infty^3] \leq c(\log p)^{3/2} \) from Lemma 7.1 yields

\[
\mathbb{E}[\|R_k^X(r)\|1\{A_k(\varepsilon)\}] \leq \frac{c(\log p)^3}{\varepsilon n^{1/2}} \left( \varepsilon \sqrt{\frac{n}{k-1}} \sqrt{\log p} + D_{k-1} \right),
\]

which holds uniformly with respect to \( r \in \mathbb{R}^p \).

**Handling the remainder on \( A_k^c(\varepsilon) \).** For this part, the idea is that for any \( r \in \mathbb{R}^p \), the quantity \( \|\nabla^3 \varphi_{\varepsilon_k}(s,r)\|_1 \) is essentially negligible when \( s \notin \partial C(r,\varepsilon) \) and \( \varepsilon \) is chosen to be sufficiently large. To this end, define the deterministic quantity

\[
b_k(\varepsilon) = \sup \left\{ \|\nabla^3 \varphi_{\varepsilon_k}(s,r)\|_1 \mid r \in \mathbb{R}^p \text{ and } s \notin \partial C(r,\varepsilon) \right\},
\]

where the supremum involves both \( s \) and \( r \). Thus, Hölder’s inequality gives

\[
\mathbb{E}[\|R_k^X(r)\|1\{A_k^c(\varepsilon)\}] \leq \frac{1}{n^{3/2}} \cdot b_k(\varepsilon) \cdot \mathbb{E}[\|X_k\|_\infty^3].
\]

Given that \( \|\nabla^3 \varphi_{\varepsilon_k}(s,r)\|_1 \) can be written down explicitly based on (15), it is straightforward to verify that if we choose

\[
\varepsilon = c\varepsilon_k \sqrt{\log (pm)}
\]

for a sufficiently large absolute constant \( c > 0 \), then \( b_k(\varepsilon) \leq \frac{c}{\varepsilon mp} \). Combining this with the fact that \( \mathbb{E}[\|X_k\|_\infty^3] \leq c(\log p)^{3/2} \) leads to the stated result. \( \square \)

**Lemma 5.2.** There is an absolute constant \( c > 0 \), such that the following holds for all \( n \geq 2 \) and \( p \geq 1 \): If the conditions of Theorem 2.1 hold, then

\[
\delta_1 \leq \frac{c(\log p)^3}{\rho^{3/2} n^{3/2}}.
\]

**Proof.** As in the proof of the previous lemma, it suffices to bound \( \sup_{r \in \mathbb{R}^p} \mathbb{E}[\|R_1^X(r)\|] \). Using the same steps as in (27) and (28), but ignoring the role of the indicator \( 1\{A_k(\varepsilon)\} \), we have

\[
\sup_{r \in \mathbb{R}^p} \mathbb{E}[\|R_1^X(r)\|] \leq \frac{c\mathbb{E}[\|X_1\|_\infty^3 \log (p)^{3/2}]}{\varepsilon n^{3/2}}.
\]

Applying the previously used bound on \( \mathbb{E}[\|X_1\|_\infty^3] \) from Lemma 7.1 completes the proof. \( \square \)

**Proof of Proposition 3.2.** By Lemma 5.2, the quantity \( \delta_1 \) is negligible in comparison to the right side of (8), and so it is enough to focus on \( \delta_2 + \cdots + \delta_{n-m} \). By Lemma 5.1, we have that for \( k = 2, \ldots, n-m \),

\[
\delta_k \leq \frac{c(\log p)^3}{\varepsilon_k n^{3/2}} \left( \varepsilon_k \log (pm) \sqrt{\frac{n}{k-1}} + D_{k-1} + \frac{1}{p^m} \right).
\]

Since we assume that \( H_1(\bar{C}), \ldots, H_{n-1}(\bar{C}) \) hold, we may derive a bound on \( \delta_k \) for each \( k = 2, \ldots, n-m \) by applying \( H_{k-1}(\bar{C}) \) to \( D_{k-1} \),

\[
\delta_k \leq \frac{c(\log p)^3}{\varepsilon_k n^{3/2}} \left( \frac{1}{(n-k)^{3/2}} + c\bar{C} \left( \frac{\nu^3}{p^m} \right) \log (k-1) \right) \frac{1}{(n-k)^{3/2} \sqrt{k-1}}.
\]
Finally, to bound the sum \( \delta_2 + \cdots + \delta_{n-m} \), observe that
\[
\sum_{k=2}^{n-m} \frac{1}{(n-k)^{3/2}k-1} \leq \frac{c\log(n)}{n^{3/2}},
\]
and
\[
\sum_{k=2}^{n-m} \frac{1}{(n-k)^{3/2}k-1} \leq \frac{c}{n^{3/2}m^{3/2}}.
\]
Combining the last few steps leads to the stated result. \( \square \)

6. Proof of Theorem 2.2. Let \( N \) be a positive integer that will be chosen later. Also, let \( Z_1, \ldots, Z_N \) be i.i.d. copies of \( Z \), and let \( Y_1, \ldots, Y_N \) be an independent sequence of i.i.d. copies of \( Y \). Due to the scale invariance of the Kolmogorov metric, we may assume without loss of generality that previous notations such as \( S \) (see also \((k)\underlying \) \((31)\)) underlying \((19)\) and \((20)\). If we account for this detail in the reasoning leading up to \((24)\), then we have the following relation for every \( k \in \{1, \ldots, N-1\} \),
\[
\delta^Z_k(r) - \delta^Y_k(r) = \mathbb{E}[Q^Z_k(r)] - \mathbb{E}[Q^Y_k(r)] + \mathbb{E}[R^Z_k(r)] - \mathbb{E}[R^Y_k(r)].
\]
The terms \( R^Z_k(r) \) and \( R^Y_k(r) \) can be handled in the same manner as before in Section 5. To handle the difference of the quadratic terms \( Q^Z_k(r) \) and \( Q^Y_k(r) \), observe that in the current context, the random vector \( \hat{r}_{k+1} \) defined in \((17)\) is independent of both \( Z_k \) and \( Y_k \), and so for \( k = 1, \ldots, N-1 \), we have
\[
\mathbb{E}[Q^Z_k(r)] - \mathbb{E}[Q^Y_k(r)] = \frac{1}{2} \mathbb{E}^{(1)} \left( \nabla^2 \varphi_{\epsilon_k}(S_{1:k-1}(Z), \hat{r}_{k+1}), N^{-1}(Z_kZ_k^\top - Y_kY_k^\top) \right).
\]
\( \square \)
the bound on \( \delta_k \) in the statement of Lemma 5.1. Apart from this, the only other modification needed is to replace the inequality \( (H_k(C)) \) in the induction hypothesis with

\[
D_k \leq C \left( \frac{(\frac{1}{\rho^{1/2}})\log(pk)^4 \log(k)}{k^{1/2}} + \frac{1}{\rho} \log(p) \log(k) \Delta \right),
\]

where \( \nu \) is absent above, because it is an absolute constant in the context of Gaussian vectors. Once these two updates are made, all of the corresponding steps in the proof of Theorem 2.1 can be repeated to show there is an absolute constant \( c > 0 \) such that the bound

\[
\mathbb{E}(X_{1j}) = 1 \text{ for all } 1 \leq j \leq p:
\]

(i) The expectation of \( \|X_1\|_\infty^3 \) satisfies

\[
\mathbb{E}(\|X_1\|_\infty^3) \leq c(\nu \log(p))^{3/2}.
\]

(ii) If \( t = c \nu \log(pm)^{1/2} n^{-1/2} \), then

\[
\mathbb{P}(\frac{1}{\sqrt{n}} \|X_1 - Y_1\|_\infty \geq t) \leq \frac{c}{n}.
\]

(iii) If \( 1 \leq m \leq n \), and the vectors \( \zeta = S_{1:n}(X) \) and \( \xi = S_{1:n-m}(X) + S_{n-m+1:n}(Y) \) are as in the proof of Lemma 3.1, then the following bound holds when \( t' = c \nu \sqrt{\frac{m}{n}} \log(pm)^{1/2} \),

\[
\mathbb{P}(\|\zeta - \xi\|_\infty \geq t') \leq \frac{c}{n}.
\]
Let $\hat{\Sigma}$ be as defined in (3), let $\Sigma = \mathbb{E}[X_1X_1^\top]$, and suppose that $n \geq \log(pn)$. Then, the event

$$\|\hat{\Sigma} - \Sigma\|_\infty \leq \frac{cn^2\log(pn)^{1/2}}{n^{1/2}}$$

holds with probability at least $1 - \frac{c}{n}$.

The next result is known as Nazarov’s Gaussian anti-concentration inequality, which originates from the paper (Nazarov, 2003), and was further elucidated by (Chernozhukov, Chetverikov and Kato, 2017b, Theorem 1).

**Lemma 7.2.** There is an absolute constant $c > 0$ such that the following holds for all $p$: Let $\xi \in \mathbb{R}^p$ be a Gaussian random vector, and suppose that $c = \min_{1 \leq j \leq p} \sqrt{\text{var}(\xi_j)}$ is positive. Then, the following inequality holds for any $t > 0$,

$$\sup_{r \in \mathbb{R}^p} \mathbb{P}(\xi \in \partial C(r, t)) \leq \frac{ct}{c} \sqrt{\log(p)},$$

where the set $\partial C(r, t)$ is defined in (7).

Here, we introduce some notation for the statement and proof of Lemma 7.3 below. For any set $A \subset \mathbb{R}^p$ and any $t > 0$, define the outer $t$-neighborhood $A^t = \{x \in \mathbb{R}^p \mid d(x, A) \leq t\}$, where $d(x, A) = \inf\{\|x - y\| \mid y \in A\}$, with $\| \cdot \|$ being any norm on $\mathbb{R}^p$. In addition, a corresponding inner $t$-neighborhood may be defined as $A^{-t} = \{x \in A \mid B(x, t) \subset A\}$, where $B(x, t) = \{y \in \mathbb{R}^p \mid \|x - y\| \leq t\}$. Although the following result is commonly used for scalar random variables, it seems to be stated less frequently in the case of random vectors.

**Lemma 7.3.** Let $\| \cdot \|$ be any norm on $\mathbb{R}^d$, and let $\zeta, \xi \in \mathbb{R}^p$ be any two random vectors. Then, the following inequality holds for any Borel set $A \subset \mathbb{R}^p$, and any $t > 0$,

$$|\mathbb{P}(\zeta \in A) - \mathbb{P}(\xi \in A)| \leq \mathbb{P}(\xi \in (A^t \setminus A^{-t})) + \mathbb{P}(\|\zeta - \xi\| \geq t).$$

**Proof.** Let $\delta = \zeta - \xi$ and observe that

$$\mathbb{P}(\xi \in A^{|\delta|}) \leq \mathbb{P}(\zeta \in A) \leq \mathbb{P}(\xi \in A^{|\delta|}).$$

This implies

$$|\mathbb{P}(\zeta \in A) - \mathbb{P}(\xi \in A)| \leq \mathbb{P}(\xi \in (A^{|\delta|} \setminus A^{-|\delta|})) \leq \mathbb{P}(\xi \in (A^t \setminus A^{-t})) + \mathbb{P}(\|\delta\| \geq t).$$

\[\blacksquare\]

**REFERENCES**


NEAR $1/\sqrt{n}$ RATES FOR CLT AND BOOTSTRAP IN HIGH DIMENSIONS


