Probability

Def1: Randomness - the outcome of an action such as rolling a die cannot be predicted with certainty

Def2: Random experiment - an experiment where several outcomes may be observed but it is not possible to predict which one it will be with certainty

Def3: Probability - for a particular outcome is the proportion of time that an outcome would occur in a long run of repetitions of a random experiment

Def4: Trial - a repetition of a random experiment

Def5: Independence - trials are independent if the outcome of one trial does not depend on the outcome of another trial
**Ex1:** Flip a fair coin 10,000 times. The possible outcomes are heads \( H \) and tails \( T \). We cannot predict at each flip which it will be. The probability of heads is \( P(H) = 0.5 \) on each flip. Heads should occur as often as tails, so we expect half the time to see heads and half the time to see tails.

**Def6:** *Objective Probability* - the probability of an outcome depends on observed frequencies of outcomes of the same type over many repetitions of a random experiment.

**Def7:** *Subjective Probability* - the probability of an outcome depends rather than data.

**Def8:** *Sample Space* - collection of all possible types of outcome of a random experiment.
**Def9:** *Equally Likely Outcomes* - have the same probability of occurring.

**Fact:** If all outcomes of a random experiment are equally likely then the probability of any particular outcome $O$ is

$$P(O) = \frac{1}{\text{number of all different outcomes}}$$

**Def10:** *Event* - is a collection of outcomes from a random experiment.

**Fact:** For a general event $A$ consisting of equally likely outcomes, the probability of $A$ is

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of all different outcomes}}$$
Note: the above calculations are for equally likely outcomes, only some experiments have equally likely outcomes.

Ex2 Flip two coins;
(a) Sample space $S = \{HH, HT, TH, TT\}$
(b) All four outcomes are equally likely. Therefore

$$P(HH) = P(HT) = P(TH) = P(TT) = .25$$

Def11: Complement $A^c$ of an event $A$ - all outcomes in $S$ that are not in $A$.

Def12: Union, $\{A \cup B\}$, of events $A$ and $B$ - all outcomes in $S$ that are in $A$ or in $B$ or both.

Def13: Intersection, $\{A \cap B\}$, of events $A$ and $B$ - all outcomes in $S$ that are in both $A$ and $B$. 
Def14: Disjoint events $A$ and $B$ - have no outcomes in common, $A$ and $B$ are also said to be mutually exclusive.

Ex3  Roll a fair die; $S = \{1, 2, 3, 4, 5, 6\}$;
   (a) $A = \{\text{even numbers}\} = \{2, 4, 6\}$;
   (b) $B = \{\text{numbers less than 4}\} = \{1, 2, 3\}$
   (c) $C = \{\text{odd numbers greater than 2}\} = \{3, 5\}$

Then

$$A^c = \{\text{all odd numbers}\} = \{1, 3, 5\}$$

$$A \cup B = \{\text{all even numbers or numbers less than 4}\}$$
   $$= \{1, 2, 3, 4, 6\}$$

$$A \cap B = \{\text{all even numbers less than 4}\} = \{2\}$$

$$A \cap C = \{2, 4, 6\} \cap \{3, 5\}$$
   $$= \{\emptyset\}$$
**Probability Rules:**

1) *Complement rule*

\[ P(A^c) = 1 - P(A) \]

2) *Additive rule for unions A, B*

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

3) *Multiplication rule for independent events A, B*

\[ P(A \cap B) = P(A) \cdot P(B) \]

4) *Probability for disjoint events A, B*

\[ P(A \cap B) = P(\phi) = 0 \]
Conditional Probability

The conditional probability of an event A given another event B is defined as

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

Note: this formula gives us a way to calculate the probability of an intersection when A and B are not independent.

**Fact:** Multiplication rule for the intersection of events A and B

\[ P(A \cap B) =^1 P(A \mid B) \cdot P(B) =^2 P(B \mid A) \cdot P(A) \]
1 You would use this form if $P(A \mid B)$ and $P(B)$ are easily calculated or known already

2 You would use this form if $P(B \mid A)$ and $P(A)$ are easily calculated or known already

**Fact:** If $A$ and $B$ are independent then

(1) $P(A \mid B) = P(A)$

(2) $P(B \mid A) = P(B)$

(3) $P(A \mid B) \cdot P(B) = P(A) \cdot P(B)$

(4) $P(B \mid A) \cdot P(A) = P(B) \cdot P(A)$

and therefore, for independent events

$$P(A \cap B) = P(A) \cdot P(B)$$