Homework 5 Solutions

3.7.41
Assume $x_0$ is the diameter of the pizza, and $X \sim N(14, 1^2)$ is the random variable for the diameter. Then $P(X \leq x_0) = 0.994$.

From the given information, we know that

$$P(X \leq x_0) = P\left(\frac{X - 14}{1} \leq x_0 - 14\right) = P(Z \leq x_0 - 14) = 0.994.$$ 

But from Table A.4, we find that $P(Z \leq 2.51) = 0.994$. Therefore, $x_0 - 14 = 2.51$ so that $x_0 = 16.51$.

3.7.46
Let $X$ have a folded normal distribution. Then,

(a) 

$$EX = \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}} xe^{-\frac{x^2}{2}} dx$$
$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} xe^{-\frac{x^2}{2}} dx$$
$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u} du \text{ by letting } u = x^2/2 \text{ so that } du = xdx$$
$$= \frac{2}{\sqrt{2\pi}} \left[ -e^{-u} \right]_{0}^{\infty} = \frac{2}{\sqrt{2\pi}}.$$

(b) 

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Let $y = x^2$. Consider

$$P(Y \leq y_0) = P(X \leq \sqrt{y_0}) = \int_{0}^{\sqrt{y_0}} \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$ 

Let $y = x^2$. Then $x = \sqrt{y}$ so that $\frac{dx}{dy} = \frac{1}{\sqrt{2}} y^{-1/2}$. Making this substitution in the above integral, we get

$$\int_{0}^{y_0} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} dy = \int_{0}^{y_0} \frac{1}{\Gamma(1/2)2^{1/2}} y^{1/2-1} e^{-y/2} dy = F(y_0)$$

where $F$ is the cdf of a $\Gamma(1/2, 2)$ distribution. Thus, $Y \sim \Gamma(1/2, 2)$.

$$\sigma_X^2 = E(X^2) - (EX)^2 = EY - (EX)^2 = 1 - \left(\frac{2}{\sqrt{2\pi}}\right)^2 = 1 - \frac{2}{\pi}.$$ 

3.7.48
This follows directly from Theorem 3.5.1 by setting $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$. To be more specific,

$$P(Z \leq z) = P\left(\frac{X - \mu}{\sigma} \leq z\right) = P(X \leq \mu + \sigma z).$$

Hence,

$$P(Z \leq z) = \int_{-\infty}^{\mu + \sigma z} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx$$
Let \( z = \frac{x-\mu}{\sigma} \). Then, \( x = \mu + \sigma z \) and hence \( J = \frac{dx}{dz} \sigma \). Thus, by the CoV theorem,

\[
P(Z \leq z) = \int_{-\infty}^{\mu+\sigma z} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx
= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x+\sigma z-\mu)^2}{2\sigma^2}} dx
= \int_{-\infty}^{\mu+\sigma z} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} dx.
\]

Thus, \( Z = \frac{X-\mu}{\sigma} \sim N(0, 1) \).

3.7.52

\[
E(1 + X) = \int_{0}^{\infty} \frac{\theta}{(1+x)^\theta} dx
= \theta \int_{0}^{\infty} (1+x)^{-\theta} dx
= \frac{\theta}{(1-\theta)(1+x)^{\theta-1}} \bigg|_{0}^{\infty}
= \frac{\theta}{\theta - 1}.
\]

Hence, \( EX = E(1 + X) - 1 = \frac{1}{\theta - 1} \).

\[
E[(1 + X)^2] = \int_{0}^{\infty} \frac{\theta}{(1+x)^{\theta-1}} dx
= \theta \int_{0}^{\infty} (1+x)^{-\theta+1} dx
= \frac{\theta}{(2-\theta)(1+x)^{\theta-2}} \bigg|_{0}^{\infty}
= \frac{\theta}{\theta - 2}.
\]

Therefore,

\[
V(X) = V(1 + X) = E[(1 + X)^2] - [E(1 + X)]^2
= \frac{\theta}{\theta - 2} - \left( \frac{\theta}{\theta - 1} \right)^2
= \frac{\theta(\theta - 1)^2 - \theta^2(\theta - 2)}{(\theta - 1)^2(\theta - 2)} = \frac{\theta}{(\theta - 1)^2(\theta - 2)}.
\]

4.8.2

(a) \( f_{Y|X}(y|x) = \frac{1}{1-x} \mathbb{I}_{(0,1-x)}(y) ; f_X(x) = 20x^3(1-x) \mathbb{I}_{(0,1)}(x) \).

Hence, \( f_{X,Y}(x,y) = 20x^3 \) for \( 0 < y < 1 - x < 1 \) (Or for \( 0 < x < 1 - y < 1 \)) Therefore, marginal density of \( Y \) is \( f_Y(y) = \int_{0}^{1} f_{X,Y}(x,y) dx = \int_{0}^{1-y} 20x^3 dx = 5(1-y)^4 \mathbb{I}_{(0,1)}(x) \)

For \( x \in (0, 1-y) \), \( f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{20x^3 \mathbb{I}_{(0,1-y)}(x)}{5(1-y)^4} = \frac{4x^3}{(1-y)^4} \mathbb{I}_{(0,1-y)}(x) \).

4.8.4

\( X = \) Number of heads in first 2 tosses \( \sim B(2,0.5) \). \( Y = \) Number of heads in last 2 tosses \( \sim B(2,0.5) \). Thus \( E(X) = E(Y) = 1, \sigma_X^2 = \sigma_Y^2 = \frac{1}{4} \).
(a) The joint p.m.f., along with the corresponding outcomes is shown in the following table

<table>
<thead>
<tr>
<th>X</th>
<th>Y →</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>P(X = x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{TTT}=1/8</td>
<td>{TTH}=1/8</td>
<td>0</td>
<td>2/8</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>{HTT}=1/8</td>
<td>{THT,HHT}=2/8</td>
<td>{THH}=1/8</td>
<td>4/8</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>{HHT}=1/8</td>
<td>{HHH}=1/8</td>
<td>2/8</td>
<td></td>
</tr>
<tr>
<td>P(Y = y)</td>
<td>2/8</td>
<td>4/8</td>
<td>2/8</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

(b) The marginal p.m.f.'s are shown in the above table. They can also be found using the fact that X and Y both follow a B(2,0.5) distribution.

(c) From the table, the conditional distribution of X given Y = 1 is obtained as following.

\[ P(X = 0|Y = 1) = \frac{1}{4}, P(X = 1|Y = 1) = \frac{2}{4}, P(X = 2|Y = 1) = \frac{1}{4} \]

**4.8.8**

(a) Marginal density of X is given by

\[
f_X(x) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} y^\alpha e^{-y/(\lambda+x)} dy \]

\[
= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{\Gamma(\alpha+1)(\frac{1}{\lambda+x})^{\alpha+1}}{\Gamma(\alpha+1)(\frac{1}{\lambda+x})^{\alpha+1}} y^{(\alpha+1)-1} e^{-y/(\lambda+x)} dy \\
= \frac{\lambda^\alpha}{\Gamma(\alpha)} \Gamma(\alpha+1) \left( \frac{1}{\lambda+x} \right)^{\alpha+1} \\
= \frac{\alpha \lambda^\alpha}{(\lambda + x)^{\alpha+1}}
\]

(b) Conditional density of Y given X = x:

\[
f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\frac{\lambda^\alpha}{\Gamma(\alpha)} y^\alpha e^{-y/(\lambda+x)}}{\frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{1}{\lambda+x}^{\alpha+1} y^{(\alpha+1)-1} e^{-y/(\lambda+x)}}
\]

\[
= \frac{(\lambda + x)^{\alpha+1}}{\alpha \Gamma(\alpha)} y^{\alpha} e^{-y/(\lambda+x)} \\
= \frac{1}{\Gamma(\alpha+1)(\frac{1}{\lambda+x})^{\alpha+1} y^{(\alpha+1)-1} e^{-y/(\lambda+x)}}
\]

So Y|X = x ~ \Gamma \left( \alpha+1, \frac{1}{\lambda+x} \right).

**4.8.10**

(a) \( f_Y(y) = \int_0^\theta \frac{1}{\theta^\alpha} e^{-y/\theta} dx = \frac{1}{\theta^\alpha} y e^{-y/\theta}; 0 < y < \infty \) which is \( \Gamma(2, \theta) \).

(b) \( f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{\lambda^\alpha}{\theta^\alpha} e^{-y/\theta}}{\frac{1}{\theta^\alpha} y e^{-y/\theta}} \mathbb{I}_{(0,y)}(x) = \frac{\lambda^\alpha}{y} \mathbb{I}_{(0,y)}(x) \) which is \( U(0, y) \).

(c) One can integrate the joint density over the region \( \{(x, y) : 0 < x < y < 2x < \infty \} \).

\[
P(Y < 2X) = \int_0^\infty \int_{0<y<2x<\infty} \frac{1}{\theta^\alpha} e^{-y/\theta} dydx = \frac{1}{\theta^\alpha} \int_0^\infty e^{-y/\theta} dy \int_0^{2x} e^{-y/\theta} dy dx
\]

\[
= \frac{1}{\theta^\alpha} \int_0^\infty -\theta e^{-y/\theta} e^{-2x/\theta} dx = \frac{1}{\theta^\alpha} \int_0^\infty (e^{-x/\theta} - e^{-2x/\theta}) dx = \frac{1}{\theta^\alpha} \left[ -\theta e^{-x/\theta} + \frac{\theta}{2} e^{-2x/\theta} \right]_0^\infty = \frac{1}{\theta^\alpha} \left[ \theta - \frac{\theta}{2} \right] = \frac{1}{2}.
\]
Alternatively, one can use the Adam’s rule to get the same answer.

\[ P(Y < 2X) = E(\mathbb{I}(Y < 2X)) = E(E(\mathbb{I}(Y < 2X)|Y)) = E(P(X > \frac{Y}{2}) \mid Y) = E(\frac{Y - Y/2}{Y}) = \frac{1}{2} \]

4.8.12

\[ P(X + Y \leq 1) = \int_{x+y\leq 1} 4xy \, dx \, dy = \int_{0}^{1} \int_{0}^{1-x} 4xy \, dx \, dy = \int_{0}^{1} 2x(1 - x)^2 = 2 \cdot \frac{\Gamma(2) \Gamma(3)}{\Gamma(5)} \quad \text{(using “the trick”)} = \frac{2 \cdot 1 \cdot 2}{4} = \frac{1}{6}. \]

4.8.14

Marginal of \( X \), \( f_X(x) = \int_{0}^{\infty} \lambda e^{-x(y + \lambda)} \, dy = \lambda e^{-\lambda x} \int_{0}^{\infty} xe^{-xy} \, dy = \lambda e^{-\lambda x} [-e^{-xy}]_{y=0}^{\infty} = \lambda e^{-\lambda x} \), so \( X \sim \text{Exp}(\lambda) \).

Conditional distribution of \( Y \) given \( X = x \): \( f_{Y \mid X}(y \mid x) = \frac{f(x, y)}{f(x)} = \frac{\lambda e^{-x(y + \lambda)}}{\lambda e^{-x}} = xe^{-xy} \), so \( Y \mid X = x \sim \text{Exp}(x) \).

4.8.16

Probability that Joe wins the bet = \( P(X > Y) = \int_{0}^{1} \int_{0}^{y} f_{X,Y}(x, y) \, dx \, dy \)

\[ = 18 \int_{0}^{1} x(1 - x) \int_{0}^{x} y^2 \, dy \, dx = 18 \int_{0}^{1} x(1 - x) \frac{x}{2} \, dx = 6 \int_{0}^{1} x^4(1 - x) \, dx = 6 \cdot \frac{\Gamma(5) \Gamma(2)}{\Gamma(7)} \quad \text{(using the “trick”)} = \frac{1}{5}. \]

4.8.18

Write \( X = (X_1, \cdots, X_n) \). Density of \( X \mid Y = y \) is given by

\[ f_{X \mid Y}(x_1, \cdots, x_n \mid y) = \Pi_{i=1}^{n} ye^{-x_i y} = y^n e^{-y \sum_{i=1}^{n} x_i}. \]

Joint density of \( X \) and \( Y \) is

\[ f_{X,Y}(x_1, \cdots, x_n, y) = y^n e^{-y \sum_{i=1}^{n} x_i} \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha - 1} e^{-y \sum_{i=1}^{n} x_i - y \lambda} = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha + n - 1} e^{-y \sum_{i=1}^{n} x_i - y \lambda}. \]

Conditional distribution of \( Y \) given \( X_1 = x_1, \cdots, X_n = x_n \) is

\[ f_{Y \mid X}(y \mid x_1, \cdots, x_n) = \frac{f_{X,Y}(x_1, \cdots, x_n, y)}{f_{X}(x_1, \cdots, x_n)} \propto y^{\alpha + n - 1} e^{-y \sum_{i=1}^{n} x_i + \lambda} \]

It can be easily identified from the form of the density that \( Y \mid X = (x_1, \cdots, x_n) \) follows a \( \Gamma \left( n + \alpha, \frac{1}{\sum_{i=1}^{n} x_i + \lambda} \right) \) distribution.