Some tests for log-concavity of life distributions

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Abstract : We consider the testing problem for log-concavity of a life distribution where the alternative is that the distribution is not log-concave. We suggest an exact test for the (restricted) class of log-concave distributions having a point mass at the left end point of the support. We prove consistency of the test which is based on the distance between logarithm of the empirical distribution function and its *Least Concave Majorant (LCM)*, and extend it to the general case. We propose yet another test based on a new transform called *Total Time since Failure (TTF)*, whose properties we study. In particular the strong uniform convergence of sample TTF to population TTF is shown under some general conditions for distributions with finite support. We show that the TTF-based test for log-concavity is consistent under the same set-up. We provide simulation-based cut-off points for both the tests and study their power properties.

Keywords : Reversed hazard rate, total time on test, total time since failure, reliability, consistent test.

1 Introduction

A distribution function F with support over $[0, \infty)$ is said to be *log-concave* if $\log F$ is a concave function, that is,

$$F(\alpha x + (1 - \alpha y)) \geq F^{\alpha}(x) F^{1-\alpha}(y) \tag{1}$$

for $\alpha \in (0,1)$ and $0 \leq x < y < \infty$. If the density exists everywhere and is denoted by f, then F is log-concave if and only if the *reversed hazard rate* function, defined by f/F, is non-increasing. The class of log-concave distributions have the following closure properties. If the random variables X and Y are independent and have (possibly different) log-concave distributions, then $X+Y$ and $\max\{X,Y\}$ have log-concave distributions (see Sengupta and Nanda, 1999). Limits of log-concave distributions are also log-concave.

Log-concave distributions play an important role in various models in econometrics and reliability. In theory of contracts, Laffont and Tirole (1988) considered the situation where the principal lacks knowledge of the relevant characteristic of an agent, but knows the distribution of this characteristic. If this distribution is log-concave, then optimal incentive contract is invertible and a separating equilibrium exists. Bagnoli and Bergstrom (1989) proposed a model to determine the cost-effectiveness of appraisals of used items when the seller knows the exact worth of the commodity but the buyer only knows its distribution. A clear decision rule on whether the seller should go for the appraisal emerges when the distribution is log-concave. The log-concave nature of a distribution happens to be a crucial assumption that ensures the existence of a separating equilibrium in a model for firms and regulators (see Baron and Myerson, 1982), and of 'efficient auctions' in a model for the analysis of auctions (see Myerson and Satterthwaite, 1983). Bergstrom and Bagnoli (1993) proposed a marriage market model where the existence of a unique equilibrium distribution of marriages by age and 'quality' of partners is ensured if the distribution of this 'quality' is log-concave. In the field of reliability, a sharp upper bound on the reliability of a unit with known mean life and log-concave life distribution was given by Sengupta and Nanda (1999). They also provided an explicit lower bound on the distribution of the number of failures (within a specified time-frame) of a system under the regime of 'perfect repair', when the failure time distribution is log-concave.

All these results are relevant when the concerned distribution is *assumed* to be log-concave. There is no readily available mechanism for checking this assumption, that is, no statistical test for the hypothesis of log-concavity of a given distribution. The purpose of the present article is to fill this void. Let $\mathcal{G}$ be the class of all log-concave distributions. Then we intend to construct a statistical test for

$$\begin{aligned} \text{null hypothesis } (\mathcal{H}_0) &: \quad F \in \mathcal{G}, \\ \text{against} \quad \text{alternative hypothesis } (\mathcal{H}_1) &: \quad F \in \mathcal{G}^c, \end{aligned} \tag{2}$$

on the basis of samples from the distribution F.

Note that if F is replaced by $1-F$ in (1), the corresponding inequality defines the class of life distributions with log-concave survival function, which is also known as the *increasing failure rate* (IFR) class. Although there are some similarities between the IFR and log-concave classes of life distributions, there is only a partial overlap between them. Another related class is that of *unimodal* distributions. A unimodal distribution on $[0, \infty)$ with mode at 0 is always log-concave. If X and Y are independent random variables with densities f and g where g is unimodal and f is itself a log-concave function, then the density of $X+Y$ is log-concave (see Steutel, 1985). Distributions having log-convex or log-concave density are log-concave.

Some tests for membership of a distribution to the unimodal and IFR classes have been proposed. The tests for unimodality include the bandwidth test (Silverman, 1981), the dip test (Hartigan and Hartigan, 1985) and the excess-mass test (Hartigan, 1987), which is equivalent to the dip test for univariate data. Tenga and Santner (1984a, 1984b) proposed a test for the hypothesis that a distribution is IFR. However, adapting this test for the log-concave class is not easy. The main difficulty in this adaptation is that while the exponential distribution lies at the boundary of the IFR and non-IFR classes, while there is no unique 'borderline' distribution in the case of the log-concave class.

We present in Section 2 a nonparametric estimator of a distribution function, under the restriction of log-concavity. Using this estimator as basis, we derive in Section 3 a consistent test for the log-concavity of a distribution, *assuming that the distribution has a point mass of a minimum size at zero.* We remove this assumption in Section 4, but the consistency of the resulting test is not established. In the two subsequent sections we develop a consistent test, based on a new transform. This transform, called the *total time since failure* (TTF) is analogous to the *total time on* *test* (TTF) transform which is well-known in the reliability literature. We provide in Section 7 approximate cut-off values for the test statistics for various levels and sample sizes, by means of simulations. We also give some estimates of power of the tests under some specific distributions which are not log-concave. All proofs are given in the appendix.

2 A nonparametric estimator of the distribution function

Let q be a real-valued function defined on an interval I on the real line. If q is bounded from above and $\mathcal{C}(q)$ is the class of all concave functions c such that $c(x) \geq q(x)$ $\forall\ x \in I$, then the *least concave majorant* (LCM) C_q of q is defined by

$$C_q(x) = \inf\{c(x) : c \in \mathcal{C}(q)\} \quad \text{for} \ \ x \in I.$$

Note that $C_q \in \mathcal{C}(q)$, that is, the infimum is attained.

A log-concave estimator of F based on n samples from F is $\exp(C_{\log F_n})$, where F_n is the empirical distribution function. Note that $\log F_n$ is a nondecreasing and piecewise constant function. The shape of the LCM of such functions is described in the following lemma.

Lemma 2.1. *Let the function $q : [a, b] \mapsto (-\infty, 0]$ be defined as*

$$q(x) = \begin{cases} v_j & \text{if } b_j \leq x < b_{j+1},\ 1 \leq j \leq n-1, \\ v_n & \text{if } x = b_n, \end{cases} \tag{3}$$

where $a = b_1 < b_2 < \cdots < b_n = b$, and $v_1 < v_2 < \cdots < v_n \leq 0$. Then

(a) C_q coincides with q over a subset of the points $b_1, \dots, b_n$, and is linear in between the points of coincidence;

(b) For any $1 \leq i < j \leq n$ let $L_{ij}(x) = v_i + (x - b_i)(v_j - v_i)/(b_j - b_i)$, $b_i < x < b_j$. Then C_q is given by

$$C_q(x) = \begin{cases} v_1 & \text{if } x = b_1 \\ \max\left\{v_j, \max\limits_{i,k:1\leq i<j<k\leq n} L_{ik}(b_j)\right\} & \text{if } x = b_j,\ 2 \leq j \leq n, \\ v_n & \text{if } x \geq b_n. \end{cases} \tag{4}$$

and by linear interpolation for x between b_j values.

The proof of Lemma 2.1 follows along the lines of those of Lemma 2.1 and Theorem 2.1 of Tenga and Santner (1984a).

The following theorem establishes the almost sure consistency of the estimator $\exp(C_{\log F_n})$ when the parent distribution is log-concave.

Theorem 2.2. *If F_n is the empirical distribution function obtained from n samples of the log-concave distribution F, which F has a point mass at some x_0, then*

(a) $\sup_{x \geq x_0} |C_{\log F_n}(x) - \log F(x)| \to 0$ *almost surely as $n \to \infty$;*

(b) $\sup_{x} |\exp(C_{\log F_n}(x)) - F(x)| \to 0$ *almost surely as $n \to \infty$.*

If F is log-concave and does not have a point mass, then for all x s.t. $F(x) > 0$, $|C_{\log F_n}(x) - \log F(x)| \to 0$ almost surely as $n \to \infty$.

Note that the estimator $C_{\log F_n}$ is *not* the nonparametric maximum likelihood estimator (MLE) of $\log F$ under the assumption of log-concavity. It can be shown using an argument similar to Barlow et al. (1972, Section 5.3) that such a constrained MLE of a log-concave distribution does not exist, unless additional restrictions are used. It can also be shown that the limit of (nonparametric) log-concave MLEs in a sequence of restricted classes converge pointwise to a piecewise linear estimator of $\log F$, as the restriction is gradually relaxed. This "log-concave MLE" (see Sengupta and Paul, 2004) is somewhat different from $C_{\log F_n}$. We use the latter estimator as the basis for the test for log-concavity discussed in the next section, because of the analytical simplicity of this estimator.

3 A test for log-concavity of distributions having mass at 0

Suppose that F_n has the empirical distribution based on n samples from the distribution F. For any step function q defined on an interval I on the real line, let

C_q be as in Section 2, and L_q be the piecewise linear function obtained from q by linear interpolation between its successive jump points. If F is indeed log-concave, then (at least for large enough n) the supremum of the difference between $C_{\log F_n}$ and $L_{\log F_n}$ should be small, whereas we would expect the difference to be appreciably large if $\log F$ deviates from concavity. In order to keep the difference well-behaved, we shall consider in this section only those distributions for which $\log F$ is bounded from below. These are distributions having a point mass at 0.

Let $\mathcal{L}_p$ be the class of distributions with support in $[0, \infty)$ and having a point mass p, with $0 < p < 1$. Notice that if F is log-concave then it can have at most one jump discontinuity, and the jump can occur only at the left end-point of its support (see Sengupta and Nanda, 1997). Thus, the distributions in $\mathcal{L}_p \cap \mathcal{G}$ do not have any discontinuity except at 0. Instead of the hypothesis (2), in this section we derive a test for

$$\begin{aligned} &\text{null hypothesis } (\mathcal{H}_0) \;:\; F \in \mathcal{L}_p \cap \mathcal{G}, \\ \text{against} \quad &\text{alternative hypothesis } (\mathcal{H}_1) \;:\; F \in \mathcal{L}_p \cap \mathcal{G}^c, \end{aligned} \tag{5}$$

on the basis of samples from the distribution F.

Let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the order statistics, $v_j = \log F_n(X_{j:n})$ and $g_j = C_{\log F_n}(X_{j:n})$. Then the test will be based on the statistic

$$d_n = \max_{1<j<n} \{(g_j - v_j) w_j\}, \tag{6}$$

where $w_1, \ldots, w_n$ are non-negative weights. It is easy to see that d_n is invariant under scale change of the samples. In the special case $w_j = 1 \; \forall j$, the statistic reduces to

$$d_n = \sup_x \{C_{\log F_n}(x) - L_{\log F_n}(x)\}. \tag{7}$$

Remark 3.1. If $X_{i:n} = \cdots = X_{j:n} < X_{j+1:n}$ for some $1 \leq i < j \leq n$ (take $X_{n+1:n}$ to be ∞), then we have $v_i = \cdots = v_j = \log(j/n)$. Consequently, $g_i = \cdots = g_j$. In the absence of ties, $v_j = \log(j/n)$.

Even though there is no 'borderline' distribution for the class $\mathcal{G}$, there is such a distribution for the class $\mathcal{L}_p \cap \mathcal{G}$, which is

$$F_p^*(x) = \begin{cases} 0 & \text{if } x < 0, \\ p^{1-x} & \text{if } 0 \le x < 1, \\ 1 & \text{if } x \ge 1. \end{cases} \tag{8}$$

Note that $\log F_p^*$ is a straight line over [0,1]. The next result shows that F_p^* can be used to obtain the 'worst-case' null distribution of d_n.

Theorem 3.2. *Let $F \in \mathcal{L}_p$ and F_p^* be as in (8). Then*

$$P_{F_p^*}(d_n \ge u) \ge P_F(d_n \ge u) \quad \forall\ u \ge 0. \tag{9}$$

The above result allows us to obtain a conservative test for log-concavity restricted to the class $\mathcal{L}_p$ for a given level $\alpha \in (0,1)$, sample size n and a fixed $p \in (0,1)$. Let $c_{\alpha,n}(p)$ be the $(1-\alpha)$ quantile of the distribution of d_n, assuming that the sample is from F_p^*. Then the test with rejection region

$$\{d_n > c_{\alpha,n}(p)\} \tag{10}$$

is a conservative test for (5) at level α.

Remark 3.3. If F is a log-convex distribution (defined by (1) with the inequality reversed) with support on $[0,a)$ for any $a > 0$ and $F \in \mathcal{L}_p$, then Theorem 3.2 holds with the inequality reversed. It follows that the test given by (10) is unbiased when the alternative hypothesis corresponds to the class of log-convex distributions with finite support and a point mass p at 0.

The exact distribution of the test statistic under the distribution F_p^* is complicated. Hence the cutoff points $c_{\alpha,n}(p)$ are found by simulation (see Section 7).

Remark 3.4. In view of Theorem 4.1 given in the next section, $c_{\alpha,n}(p_*)$ is a conservative cut-off for d_n if $p_* > p$. Thus, there is no need to know the point mass p at zero exactly; one can work with a lower also.

We now prove the consistency of the test under two different sets of conditions on the weights.

Theorem 3.5. *The test with critical region (10) is consistent for the testing problem (5) if either of the following conditions hold.*

(a) The weights $w_1, \ldots, w_n$ satisfy $\underline{w} \leq w_j \leq \overline{w} \quad \forall\, j, n$ for some $0 < \underline{w} \leq \overline{w} < \infty$.

(b) The weights $w_1, \ldots, w_n$ are such that $w_j = w(j/n)$ where $w : [0,1] \mapsto \mathbb{R}$ is a continuous function such that $w(x) > 0\ \forall\, x > 0$.

4 A conservative test for log-concavity

We now return to the main testing problem (2). When F does not have a point mass at 0, $\lim_{x\to 0} \log F(x) = -\infty$. As the probability mass in a right neighbourhood of 0 is small, for any given sample size there may be shortage of data to detect departure from log-concavity in this region. The problem is simplified if we have the following information about a quantile of F: *there are small positive numbers x_0 and p_0 such that $F(x_0) \geq p_0$*. We can subtract x_0 from all the sample values and equate the negative values to zero. The modified sample has the distribution F_0 given by

$$F_0(x) = F(x + x_0),$$

and has a point mass of at least p_0 at 0. Thus, the statistic d_n with conservative cutoff $c_{\alpha,n}(p_0)$ can be used (see Remark 3.4) to test the log-concavity of F_0. This is equivalent to testing the log-concavity of F in the interval $[x_0, \infty)$.

We now assume that no such auxiliary information is available along with the data. Using the test statistic d_n derived in Section 3 as a starting point, we study the behaviour of the cut-off $c_{\alpha,n}(p)$ as p goes to zero.

Theorem 4.1. *For every fixed n and $\alpha \in (0,1)$, as p decreases to zero, $c_{\alpha,n}(p)$ increases monotonically to a finite limit.*

Let

$$c_{\alpha,n} = \lim_{p\to 0} c_{\alpha,n}(p). \tag{11}$$

For a given level α, we propose to test (2) via the rejection region

$$\{d_n > c_{\alpha,n}\} \tag{12}$$

If the null distribution is in $\mathcal{L}_p \cap \mathcal{G}$, then the cutoff $c_{\alpha,n}(p)$ is adequate for achieving the level α. Increasing the cut-off to $c_{\alpha,n}$ would mean that the rejection probability is smaller, that is, the test is conservative. The following result shows that the above test is a conservative one at level α, even if there is no point mass at zero.

Theorem 4.2. *Let the statistic d_n correspond to a sample of size n from any log-concave distribution F. Then*

$$P(d_n > c_{\alpha,n}) \leq \alpha. \tag{13}$$

When F does not have a point mass at zero, the lack of uniform almost sure convergence of $\log F_n(x)$ to $\log F(x)$ makes it difficult to establish that $\lim_{n\to\infty} c_{\alpha,n} = 0$. This is in contrast to the case of $F \in \mathcal{L}_p$ where we proved that $\lim_{n\to\infty} c_{\alpha,n}(p) = 0$ for every fixed $p \in (0,1)$. Thus, consistency of the test for general F remains an open question.

5 Total Time since Failure (TTF)

The *Total Time on Test (TTT)* transform of a life distribution is found to be very useful in reliability, particularly in the study of monotone ageing. We define an analogous transform with a view to deriving another test of log-concavity of life distributions which would be demonstrably consistent. In order to do this, we restrict attention to the sub-class of *distributions with bounded support contained in* $[0,\infty)$, which we denote by $\mathcal{L}_B$. We use the notation $\mathcal{L}$ to represent the class of all distributions with support in $[0,\infty)$.

In the discussion to follow, we define the inverse of any function $g : \mathbb{R} \mapsto \mathbb{R}$ as

$$g^{-1}(t) = \inf\{y : g(y) \geq t\} \quad \text{for} \quad t \in \mathbb{R}. \tag{14}$$

Definition 5.1. If $F \in \mathcal{L}_B$ with support $[a,b]$ where a and b are real numbers, then the *Total Time since Failure (TTF)* transform of F is given by

$$T_F^{-1}(t) = \frac{\int_a^{F^{-1}(t)} F(u)du}{\int_a^b F(u)du}. \tag{15}$$

The notation T_F^{-1} is used only to simplify our later notations, where inverse of this function will be used. The following lemma describes some useful properties of the TTF transform.

Lemma 5.2. *Let F be a distribution in $\mathcal{L}_B$, having support $[a, b]$.*

(a) $T_F^{-1} : [0,1] \mapsto [0,1]$ is a left-continuous and monotone increasing function.

(b) F has a discontinuity at x_0 ($x_0 > a$) if and only if T_F^{-1} has zero slope in a left-neighbourhood of $F(x_0)$.

(c) T_F^{-1} has a jump discontinuity at u_0 ($0 < u_0 < 1$) if and only if F has zero slope in a right-neighbourhood of $F^{-1}(u_0)$.

(d) F is log-concave if and only if T_F^{-1} is convex.

If F_n is the empirical distribution function for a random sample of size n drawn from a distribution in $\mathcal{L}$ with support $[a, b]$, then the *sample TTF* is defined by

$$T_n^{-1}(t) = \frac{\int_a^{F_n^{-1}(t)} F_n(u)du}{\int_a^{F_n^{-1}(1)} F_n(u)du}. \tag{16}$$

Remark 5.3. The expression for T_n^{-1} can be written explicitly in terms of the order statistics $(X_{1:n}, \ldots, X_{n,n})$. Let a be the left end-point of the support of F. Let $S_{k;n} = T_n^{-1}\left(\frac{k}{n}\right)$. Then if $X_{n:n} > a$,

$$S_{k;n} = \frac{\sum_{j=1}^{k}(X_{k:n} - X_{j:n})}{\sum_{j=1}^{n}(X_{n:n} - X_{j:n})} \quad \text{for} \quad k = 0, 1, \ldots, n, \tag{17}$$

so that $S_{0;n} \equiv S_{1;n} \equiv 0$ and $S_{n;n} \equiv 1$. If $X_{n:n} = a$, then we define $S_{k;n} = 0$ for $k = 0, 1, \ldots, n-1$ and $S_{n;n} = 1$. This convention makes the mapping

$$(X_{1:n}, \ldots, X_{n:n}) \mapsto (S_{0;n}, \ldots, S_{n;n})$$

continuous. T_n^{-1} is a left continuous step function with values $S_{k;n}$ at the points k/n. It is convenient, however, to consider the function $\widetilde{T}_n^{-1}$ whose graph is obtained by linear interpolation between the successive points $(k/n, S_{k;n})$. In particular, viewed as a function of t, $\widetilde{T}_n^{-1}(t)$ is a continuous distribution function for every fixed sample realization.

The next result shows uniform almost sure convergence of the sample TTF (and its linearized version) to the population TTF, under certain conditions.

Theorem 5.4. *Suppose $F \in \mathcal{L}_B$ is such that F^{-1} is continuous on $[0,1]$ (F is strictly increasing on its support, which is an interval). Then,*

$$(a) \sup_{t\in[0,1]} |T_n^{-1}(t) - T_F^{-1}(t)| \quad \to \quad 0 \quad \textit{almost surely} \quad \textit{as} \quad n \to \infty, \tag{18}$$

$$(b) \sup_{t\in[0,1]} |\widetilde{T}_n^{-1}(t) - T_F^{-1}(t)| \quad \to \quad 0 \quad \textit{almost surely} \quad \textit{as} \quad n \to \infty. \tag{19}$$

Corollary 5.5. *If $F \in \mathcal{L}_B$ is log-concave, then*

$$(a) \sup_{t\in[0,1]} |T_n^{-1}(t) - G_{T_n^{-1}}(t)| \quad \to \quad 0 \quad \textit{almost surely} \quad \textit{as} \quad n \to \infty, \tag{20}$$

$$(b) \sup_{t\in[0,1]} |\widetilde{T}_n^{-1}(t) - G_{\widetilde{T}_n^{-1}}(t)| \quad \to \quad 0 \quad \textit{almost surely} \quad \textit{as} \quad n \to \infty. \tag{21}$$

where G_q stands for the Greatest Convex Minorant of a function q, and is defined in a manner similar to the Least Concave Majorant, C_q.

Remark 5.6. If the upper end-point of the support of the distribution is not known, then a known upper bound of this end-point can be used. Theorem 5.4 and Corollary 5.5 go through even with this modification.

The statistic of part (b) of Corollary 5.5 is an empirical measure of departure of F from log-concavity. An analogous measure can be obtained via an inverse of the TTF transform. Let the "lower inverse" of any monotone nondecreasing function $g : [0,1] \to [0,1]$ be defined by

$$g^{-L}(x) = \sup\{t \in [0,1] : g(t) \le x\} \quad \text{for} \quad x \in [0,1]. \tag{22}$$

Let us define the inverse population TTF for $F \in \mathcal{L}_B$ by

$$T_F(x) = \begin{cases} 0 & \text{if } x < 0, \\ (T_F^{-1})^{-L}(x) & \text{if } x \in [0,1], \\ 1 & \text{if } x > 1. \end{cases} \tag{23}$$

Likewise, let the corresponding inverse of the linearized sample TTF be

$$\widetilde{T}_n(x) = \begin{cases} 0 & \text{if } x < 0, \\ (\widetilde{T}_n^{-1})^{-L}(x) & \text{if } x \in [0,1], \\ 1 & \text{if } x > 1. \end{cases} \tag{24}$$

For every fixed sample realization, $\widetilde{T}_n$ is a distribution function with support $[0,1]$. It has a point mass at zero and is continuous everywhere else. Actually, $\widetilde{T}_n$ is the linear interpolation between the successive points $(S_{k;n}, k/n)$ for $k = 1, \dots, n$.

Remark 5.7. $F \in \mathcal{L}_B$ is log-concave if and only if T_F is concave on $[0,1]$. In such a case, T_F coincides with its LCM, C_{T_F}.

Theorem 5.8. *Let $F \in \mathcal{L}_B$ be such that F^{-1} is continuous on $[0,1]$. Also, suppose that F has at most one point mass, and if it exists it is at 0. Then,*

$$(a) \sup_{t\in[0,1]} |\widetilde{T}_n(t) - T_F(t)| \quad \to \quad 0 \quad \textit{almost surely} \quad \textit{as} \quad n \to \infty, \tag{25}$$

$$(b) \sup_{t\in[0,1]} |\widetilde{T}_n(t) - C_{\widetilde{T}_n}(t)| \quad \to \quad 0 \quad \textit{almost surely} \quad \textit{as} \quad n \to \infty, \tag{26}$$

if F is also log-concave. If F has point mass at any point other than 0, then F is not log-concave and so (b) fails to hold but (a) holds with the uniform convergence replaced by pointwise convergence.

6 A test for log-concavity based on sample TTF

We use a weighted version of the statistic of part (b) of Theorem 5.8 to test for the log-concavity of a distribution. Let $b_j = S_{j;n}$, $v_j = \widetilde{T}_n(b_j)$ and $\widetilde{g}_j = C_{\widetilde{T}_n}(b_j)$. Note that if there are no ties in the observations then $v_j = j/n$. For a given sequence of nonnegative weights $\{w_j : j = 1, \dots, n\}$ we define the test statistic by

$$t_n = \max_{1<j<n} \{(\widetilde{g}_j - v_j) w_j\}, \tag{27}$$

where $w_1, \dots, w_n$ are nonnegative weights.

The statistic t_n is invariant under scale change of the samples, as $S_{1;n}, \dots, S_{n;n}$ are scale invariant.

As in Section 3, we first consider the class $\mathcal{L}_p \cap \mathcal{G}$ of log-concave distribution having point mass p at 0. Once again, the worst-case distribution happens to be F_p^* defined by (8).

Theorem 6.1. *Let $F \in \mathcal{L}_p \cap \mathcal{G}$ and F_p^* be as in (8). Then*

$$P_{F_p^*}(t_n \geq u) \geq P_F(t_n \geq u) \quad \forall\, u \geq 0. \tag{28}$$

Now define $\tau_{\alpha,n}(p)$ to be the $(1-\alpha)$ quantile of the distribution of t_n, assuming that the sample is from F_p^*. This quantile is an increasing function of p.

Theorem 6.2. *For every fixed n and $\alpha \in (0,1)$, as p decreases to zero, $\tau_{\alpha,n}(p)$ increases monotonically to a finite limit.*

Let

$$\tau_{\alpha,n} = \lim_{p \to 0} \tau_{\alpha,n}(p). \tag{29}$$

As the restriction of a point mass at 0 is removed, the following proposition allows us to use $\tau_{\alpha,n}$ as a conservative cutoff for a level α test for log-concavity.

Theorem 6.3. *Let the statistic t_n correspond to a sample of size n from any log-concave distribution F. Then*

$$P(t_n > \tau_{\alpha,n}) \leq \alpha. \tag{30}$$

Unlike in the cutoff $c_{\alpha,n}$ used in Section 4, we can identify $\tau_{\alpha,n}$ as the $(1-\alpha)$ quantile of a particular distribution. Let $E_{1:n} < E_{2:n} < \ldots < E_{n:n}$ denote the order statistics corresponding to a random sample of size n from a unit exponential distribution. Define for $k = 1, \ldots, n$,

$$Z_{k;n} = \frac{\sum_{j=n-k+1}^{n}(E_{j:n} - E_{n-k+1:n})}{\sum_{j=1}^{n}(E_{j:n} - E_{1:n})} \tag{31}$$

Lemma 6.4. *Let $\mathbf{Z} = (Z_{1;n}, \ldots, Z_{n;n})$ be defined by (31) and let $\mathbf{S}_p = (S_{1;n}, \ldots, S_{n;n})$ be such that $S_{k;n}$ are defined by (17) where $\mathbf{X}$ is a random sample from F_p^*. Then,*

$$As\ p \downarrow 0, \quad \mathbf{S}_p \stackrel{\mathcal{D}}{\Longrightarrow} \mathbf{Z} \tag{32}$$

Let $\mathbf{Z}$ be is as in Lemma 6.4. Let V_n be the function whose graph in the range $0 \le x \le 1$ is obtained by linearly interpolating successive points of the set $(Z_{k;n}, k/n)$ for $k = 1, \ldots, n$. Also let $V_n(x) = 0$ for $x < 0$ and $V_n(x) = 1$ for $x > 1$. Let $v_j = V_n(Z_{j;n})$ and $h_j = C_{V_n}(Z_{j;n})$ for $j = 1, \ldots, n$. Then, for $\{w_j : j = 1, \ldots, n\}$ as in the definition of t_n, we define

$$\widetilde{t}_n = \max_{1<j<n} \{(h_j - v_j) w_j\} \tag{33}$$

Theorem 6.5. *Under the above set-up, $\tau_{\alpha,n}$ is the $(1-\alpha)$ quantile of $\widetilde{t}_n$.*

Let the statistic t_n be computed from a sample of size n from the distribution $F \in \mathcal{L}_B$. Then a conservative test for the null hypothesis $H_0 : F \in \mathcal{G}$ against the alternative $H_1 : F \in \mathcal{G}^c$, given by the rejection region

$$t_n > \tau_{\alpha,n} \tag{34}$$

has size at most equal to α.

The restriction that F should belong to $\mathcal{L}_B$ is not necessary to perform the test, since the statistic t_n can be defined for any $F \in \mathcal{L}$. However, without this restriction the function T_F is not properly defined and the convergence result (25) does not hold. Subject to this restriction, we establish the consistency of the proposed test, via the next theorem.

Theorem 6.6. *Let F be restricted to the class of distributions as in Theorem 5.8(a) and suppose the weights satisfy condition (a) of Theorem 3.5. Then the test having rejection region (34) is consistent.*

7 Simulation results

The cut-off value for the test statistic d_n for the level α is $c_{\alpha,n}$, which is the limit of $c_{\alpha,n}(p)$ as $p \to \infty$. Table 1 gives cut-off points for $\alpha = 0.05$ only, though the pattern of the cut-off values for different p and n holds generally for other values of α. The weights $w_1, \ldots, w_n$ are taken as 1.

TABLE 1. CUT-OFF VALUES $c_{\alpha,n}(p)$ AND $c_{\alpha,n}$ FOR $\alpha = .05$, $w_1 = \cdots = w_n = 1$

No. of samples	Sample size (n)	Values of $c_{\alpha,n}(p)$ for $p =$ 0.1	0.01	0.001	0.0001	0.00001	Approximate cut-off ($c_{\alpha,n}$)
50000	5	0.57733	0.63290	0.64732	0.64220	0.64398	0.65
50000	6	0.61311	0.70734	0.71729	0.71967	0.71590	0.72
50000	7	0.65054	0.75592	0.77152	0.77325	0.77350	0.78
50000	8	0.66467	0.79944	0.81703	0.81858	0.82153	0.83
50000	9	0.67445	0.83058	0.85927	0.86336	0.85895	0.87
50000	10	0.68175	0.85915	0.88518	0.89514	0.90451	0.91
50000	15	0.67707	0.93439	1.00430	1.01422	1.01057	1.02
50000	20	0.65172	0.98292	1.07420	1.08627	1.08364	1.09
50000	30	0.59803	1.01203	1.16183	1.17789	1.18222	1.19
40000	50	0.50458	0.99065	1.24960	1.29955	1.30394	1.31
30000	100	0.38566	0.90610	1.29877	1.42328	1.42458	1.43
30000	200	0.28506	0.78251	1.29405	1.51659	1.52930	1.53

The first column represents the number of simulation runs on the basis of which the reported values are determined. The values of p (point mass at zero) are taken to be 0.1, 0.01, 0.001, 0.0001 and 0.00001. The last column gives the smallest number, up to the second place of decimal, which is larger than $c_{\alpha,n}(p)$ for all these values of p, and may be regarded as an approximate value of $c_{\alpha,n}$.

TABLE 2. CUT-OFF VALUES $c_{\alpha,n}(p)$ AND $c_{\alpha,n}$ FOR $\alpha = .05$, $w_j = \min(1, (-1/\log(j/n)))$ FOR $j = 1, ..., n$.

No. of samples	Sample size (n)	Values of $c_{\alpha,n}(p)$ for $p =$ 0.1	0.01	0.001	0.0001	0.00001	0.000001	Approximate cut-off ($c_{\alpha,n}$)
50000	5	0.57694	0.63873	0.64652	0.64442	0.64591	0.64364	0.65
50000	10	0.54993	0.63378	0.65220	0.65367	0.65569	0.65277	0.66
50000	20	0.47559	0.57675	0.60723	0.60919	0.60839	0.60856	0.61
50000	30	0.42440	0.52503	0.56965	0.57688	0.57620	0.57573	0.58
40000	50	0.35845	0.45755	0.51905	0.53169	0.53413	0.53227	0.54
30000	100	0.27950	0.36956	0.44560	0.47883	0.47928	0.47840	0.48
20000	500	0.14199	0.20708	0.28450	0.36488	0.37383	0.37096	0.38
10000	2000	0.07532	0.11591	0.17775	0.25476	0.29373	0.29885	0.30

Table 2 gives the cut-off values $c_{\alpha,n}(p)$ and $c_{\alpha,n}$ for weights $w_j = \min(1, (-1/\log(\frac{j}{n})))$ for $j = 1, ..., n$, corresponding to $\alpha = .05$. Note that, for fixed n, the weights are all positive, increasing as j increases, and bounded above. Also as $n \to \infty$, $w(j) \to 0$ for all fixed j. Further, this particular weighting scheme forces the test statistic to take values between 0 and 1. The resulting cut-off values are smaller and show a decreasing trend (with increasing n) more clearly.

The conservative cut-off values of d_n for uniform weights and different levels are summarized in Table 3.

TABLE 3. CUT-OFF VALUES $c_{\alpha,n}$ FOR $w_1 = \cdots = w_n = 1$

Sample size (n)	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
5	0.79	0.65	0.56
10	1.12	0.91	0.78
20	1.39	1.09	0.94
50	1.68	1.31	1.11
100	1.85	1.43	1.22

The cut-off values for the statistic t_n are given in Table 4. Evidently $\tau_{\alpha,n}$ decreases with increasing n, as the TTF approaches the limiting curve, namely the diagonal.

TABLE 4. CUT-OFF VALUES $\tau_{\alpha,n}$

Sample size (n)	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
10	0.388	0.313	0.275
20	0.309	0.254	0.227
50	0.214	0.179	0.161
100	0.158	0.132	0.120
200	0.116	0.096	0.088

The power of the tests d_n and t_n were simulated for a piecewise exponential distribution. The graph of $\log F$ for this continuous non-log-concave distribution shown in Figure 1. The samples of this distribution can be described as

$$Z = \begin{cases} X & \text{if } X \le x_0, \\ x_0 + Y & \text{if } X > x_0, \end{cases}$$

where $X \sim \exp(\lambda_1)$ and $Y \sim \exp(\lambda_2)$.

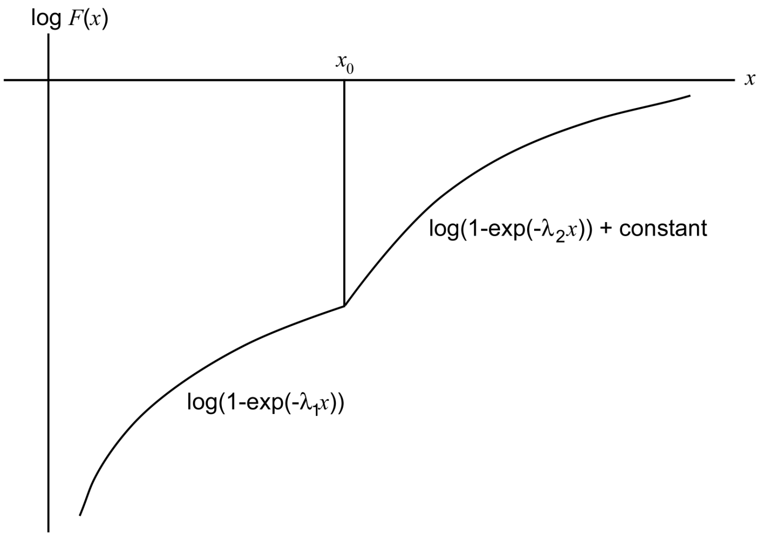

Figure 1. Non-log-concave distribution used for power comparison of d_n and t_n.

Table 5. Empirical power of d_n for piecewise exponential distribution, $\lambda_1 = 1$, $\lambda_2 = 20$, $\alpha = 0.05$. (No. of samples used in power calculation = 100000)

x_0	Empirical power $n = 20, c_{\alpha,n} = 1.09$	Empirical power $n = 50, c_{\alpha,n} = 1.31$
0.05	0.0689	0.0118
0.1	0.0833	0.0906
0.2	0.1169	0.0257
0.3	0.0746	0.0044
0.5	0.0165	0.0001

Table 6. Empirical power of t_n for rate changing exponential distribution, $\lambda_1 = 1$, $\lambda_2 = 20$, $\alpha = 0.05$. (No. of samples used in power calculation = 100000)

x_0	Empirical power $n = 20, \tau_{\alpha,n} = 0.254$	Empirical power $n = 50, \tau_{\alpha,n} = 0.179$
0.2	0.1277	0.5910
0.5	0.4876	0.9395
0.8	0.3753	0.8422
1.0	0.2530	0.6801
1.5	0.0549	0.1788
2.0	0.0070	0.0149

We chose the following values of the parameters: $\lambda_1 = 1$, $\lambda_2 = 20$ and $x_0 = .2, 1$. Table 5 shows the simulated power of d_n for sample size 50. Table 6 shows the simulated power of t_n for the same sample size. The power is found to be rather low for small sample sizes.

We next consider an empirically weighted version of the test statistics d_n and t_n. The weights that we use are a crude version of the density estimates of the observations. More specifically, given n observations from a distribution, (we want to test if it is log-concave) we generate a histogram of the observations with equal bin-width and number of bins approximately $\sqrt{n}$. Denoting this histogram-based density estimate by $\hat{f}_n(x)$, we define our weight sequence $\{w_i^n : i = 1, \ldots, n\}$ as $w_i^n = \hat{f}_n(X_{i:n})$. Using these weights we can compute the weighted versions of the two test statistics d_n and t_n. Further, we can simulate samples of same size n, from the borderline (or limiting borderline) null distributions and using this compute the test statistics using the weight sequence $\{w_i^n\}_{i=1}^n$. With sufficiently large number of such simulated samples we can compute an approximate cutoff value for a level α test for any $0 < \alpha < 1$. The Tables 7-8 give an idea about the power properties of the tests, for the limited class of alternative distributions we considered before. It should be remembered however, that in case of the first test (with test statistic d_n, i.e., the one based on the supremum (weighted) difference between $\log F_n$ and its LCM) there is no borderline distribution if there is no point mass at zero. Also, we have noted that the convergence of the cutoff value for a fixed level α test, as p (point mass at zero) decreases to zero, slows down as n increases. This fact and the large amount of computations necessary to get a single limiting conservative cutoff value forced us to choose a very small but fixed p for our simulation study. However, we let p decrease for an increased sample size in accordance with the observations made above.

The power of the empirically weighted test statistic seems to be much better for the TTF-based test (with test statistic t_n), as compared to its unweighted version. The reason is that the weighted test puts more mass at regions where we are most likely to see a discrepancy from log-concavity. However, the performance of the test

d_n does not improve much when this weighting scheme is used.

It seems that the power in general has a regular behaviour, in that it first increases and then decreases with increased value of x_0 or the change-point in the changed rate exponential alternative, when the rate parameters are held fixed. What is also interesting is that the cutoff values (now random, since the weights are random quantities, dependent on the data), also obey a pattern that their standard deviation is roughly proportional to their means. Comparing with Table 5 one can see an improvement in power of the test when these empirical weights are used.

Table 7. Empirical power of empirically weighted d_n for piecewise exponential distribution, $\lambda_1 = 1$, $\lambda_2 = 20$, $n = 50$, $\alpha = 0.05$ (No. of samples used in power calculation = 2000, No. samples drawn per cutoff computation = 1000)

p	x_0	Mean(cut-off)	s.d.(cut-off)	Empirical power
	0.3	0.277	0.087	0.488
0.00005	0.5	0.216	0.055	0.542
	0.7	0.197	0.047	0.282
	1.0	0.224	0.058	0.066
	1.2	0.250	0.064	0.015

Table 8. Empirical power of empirically weighted t_n for piecewise exponential distribution, $\lambda_1 = 1$, $\lambda_2 = 20$, $n = 50$, $\alpha = 0.05$ (No. of samples used in power calculation = 2000, No. samples drawn per cutoff computation = 1000)

x_0	Mean(cut-off)	s.d.(cut-off)	Empirical power
0.3	0.079	0.018	0.720
0.5	0.072	0.018	0.910
0.7	0.061	0.016	0.940
1.0	0.047	0.013	0.854
1.2	0.041	0.009	0.725
1.5	0.039	0.007	0.372

Appendix : Proofs

Proof of Theorem 2.2.

We consider two cases separately: F has a point mass at 0, and F does not have a point mass at 0.

Case I. F has a point mass p at 0.

In this case, by Lemma A.4 (given below), we have

$$P(\sup_x |\log F_n(x) - \log F(x)| \to 0) = 1.$$

So, given any $\epsilon > 0$,

$$\lim_{m\to\infty} P(\sup_x |\log F_n(x) - \log F(x)| \leq \epsilon \;\forall\; n \geq m) = 1.$$

So, since $\log F(x) + \epsilon$ is a concave function, with probability tending towards 1, as $m \to \infty$, we have uniformly over $x \geq 0$,

$$\log F(x) - \epsilon \leq \log F_n(x) \leq C_{\log F_n}(x) \leq \log F(x) + \epsilon, \quad \forall\; n \geq m.$$

This implies,

$$\begin{aligned}
&\lim_{m\to\infty} P(\sup_x |C_{\log F_n}(x) - \log F(x)| \leq \epsilon \;\forall\; n \geq m) \\
&\quad \geq \lim_{m\to\infty} P(\sup_x |\log F_n(x) - \log F(x)| \leq \epsilon \;\forall\; n \geq m) = 1.
\end{aligned}$$

In other words,

$$\sup_x |C_{\log F_n}(x) - \log F(x)| \to 0, \text{ almost surely as } n \to \infty.$$

Case II. F does not have a point mass at 0.

In this case we shall show only the pointwise almost sure convergence. Fix any $x_0 > 0$. For $0 < K < x_0$, let us define, for all $\omega \in \Omega$,

$$y_n^K(\omega) = \inf\{x \geq K : C_{\log F_n}(x) = \log F_n(x)\}.$$

Note that for every fixed K, $y_n^K(\omega) = X_{i:n}(\omega)$ for some i depending on K and ω. Since $F(K) > 0$, by Lemma A.4, given any $\epsilon > 0$, for a.a. ω, there exists $N_\epsilon(\omega)$, such that, if $n \geq N_\epsilon(\omega)$, then $\sup_{x \geq K} |\log F_n(x) - \log F(x)| \leq \epsilon$. Since $\log F(x) + \epsilon$ is concave, this implies, in particular, that for a.a. ω, if $n \geq N_\epsilon(\omega)$, then for all $x \geq y_n^K(\omega)$,

$$\log F(x) - \epsilon \leq \log F_n(x) \leq C_{\log F_n}(x) \leq \log F(x) + \epsilon.$$

This follows since the restriction of $C_{\log F_n}$ on $[y_n^K, \infty)$ is the LCM of the restriction of $\log F_n$ on $[y_n^K, \infty)$ (easy to check using the definition of y_n^K and Lemma 2.1).

We now aim to show that:

$\exists$ a $0 < K^* < x_0$, and $K^* < \eta_{K^*} < x_0$ such that for a.a. ω,

$$\limsup_{n \to \infty} y_n^{K^*}(\omega) < \eta_{K^*}.$$

So, in particular, given $\epsilon > 0$, for a.a. ω, $\exists$ $N'_\epsilon(\omega) < \infty$ such that, for all $n \geq N'_\epsilon(\omega)$, $y_n^{K^*}(\omega) < x_0$ and so, for all $x \geq x_0$,

$$|C_{\log F_n}(x) - \log F(x)| \leq |\log F_n(x) - \log F(x)| + \epsilon \leq 2\epsilon.$$

This will prove the strong uniform convergence of $C_{\log F_n}(x)$ to $\log F(x)$ on $[x_0, \infty)$.

Fix $r > 0$ (small) and consider the set of $K > 0$ such that $\log F(K) + r < 0$. Then define $\xi_{K,r}$ as the smallest x-coordinate where a straight line passing through $(0, \log F(K) + r)$ touches $\log F$. Notice that since F is strictly concave and does not have a point mass at 0, $\xi_{K,r}$ is well defined and it decreases to 0 for a.a. ω as $K \downarrow 0$, (for every fixed $r > 0$). It is also important to notice that $\xi_{K,r} > K$ for all $K>0$. Hence, we can find a $K^*>0$ such that $\xi_{K^*,r}<x_0$ (see Figure 2). Since we fix $r>0$ once for all (*it may depend upon* x_0), we can drop the explicit reference to r from now on.

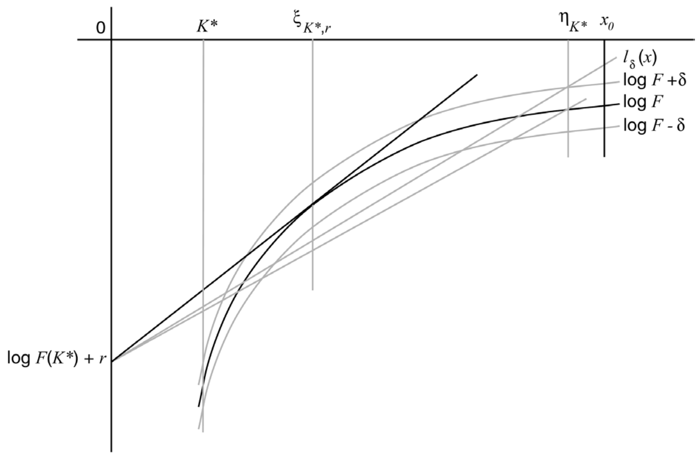

Figure 2. Construction used in the Proof of Theorem 2.2.

Let us pick an η_{K^*} such that $\xi_{K^*,r} < \eta_{K^*} < x_0$. We plan to show that $y_n^{K^*}(\omega) < \eta_{K^*}(\omega)$ for n sufficiently large (for a.a. ω). By the strict concavity of $\log F$, it follows that the slope of the line touching $\log F$ at ξ_{K^*} and passing through $(0, \log F(K^*)+r)$ is strictly bigger than the slope of the line joining the points $(0, \log F(K^*)+r)$ and $(\eta_{K^*}, \log F(\eta_{K^*}))$. Hence we can choose a $0 < \delta < r$ such that if $l_\delta(x)$ denotes the line joining the points $(0, \log F(K^*)+r)$ and $(\eta_{K^*}, \log F(\eta_{K^*})+\delta)$, then

$$l_\delta(\xi_{K^*}) < \log F(\xi_{K^*}) - \delta.$$

On the other hand, note that we can choose n large enough (depending on ω) so that

$$\sup_{x \geq K^*} |\log F_n(x) - \log F(x)| < \delta,$$

implying

$$\log F_n(y_n^{K^*}(\omega)) < \log F(y_n^{K^*}(\omega)) + \delta.$$

Therefore, if $y_n^{K^*}(\omega) \geq \eta_{K^*}$ infinitely often, then the line joining $(y_n^{K^*}(\omega), \log F_n(y_n^{K^*}(\omega)))$ and $(0, \log F(K^*)+r)$ must intersect the curve $\log F(\cdot) - \delta$ for infinitely many n, by the observation made in the previous paragraph.

Again since for a.a. ω, $\log F_n(K^*) \to \log F(K^*)$ and for all $x \leq K^*$,

$$\log F_n(x) \leq \log F_n(K^*) < \log F(K^*) + r,$$

for all sufficiently large n, so the above statement holds true even if we replace the point $(0, \log F(K^*)+r)$ by the point $(x, \log F_n(x))$ for any $0 < x \leq K^*$ for sufficiently large n.

In view of Lemma A.1 (given below), for every $\epsilon > 0$, $\exists\ C_\epsilon$ with $0 < C_\epsilon < K^*$, such that the set

$$A_\epsilon = \{\omega : \liminf x_n^*(\omega) > C_\epsilon\}$$

has probability greater than $1 - \epsilon$. Now, for a.a. ω,

$$\sup_{x \geq C_\epsilon} |\log F_n(x) - \log F(x)| < \delta \tag{35}$$

for sufficiently large n. Now fix an $\omega \in A_\epsilon$ such that (35) holds. Notice that by definition of LCM, the line segment joining $(x_n^*(\omega), \log F_n(x_n^*(\omega)))$ and $(y_n^{K^*}(\omega), \log F_n(y_n^{K^*}(\omega)))$

(call it $\widetilde{L}$) will be part of $C_{\log F_n}$ and hence should lie above $\log F_n$. However, by the observation made above, $\widetilde{L}$ intersects $\log F - \delta$ infinitely often, and this is a contradiction to (35).

Since $\epsilon > 0$ is arbitrary, this contradiction proves the result. $\square$

Lemma A.1. *Let*

$$x_n^*(\omega) = \sup\{x < K^* : C_{\log F_n}(x) = \log F_n(x)\}.$$

Then, for a.a. ω,

$$\liminf_{n\to\infty} x_n^*(\omega) > 0.$$

Proof. Suppose $\exists$ a set A of positive probability such that for all $\omega \in A$, $\liminf x_n^*(\omega) = 0$. By definition, $x_n^*(\omega) = X_{k_n:n}(\omega)$ for some $1 \le k_n(\omega) \le n$. Since F does not have a point mass at zero, $\liminf x_n^*(\omega) = 0$ implies that a subsequence of $n^{-1}k_n(\omega)$ converges to zero. To avoid messy notations, we assume without loss of generality that the original sequence itself converges to zero. This means, of course, that $\log F_n(x_n^*(\omega)) \to -\infty$. Now, for any M such that $0 < M < K^*$,

$$\sup_{x\ge M} |\log F_n(x) - \log F(x)| < \delta \tag{36}$$

almost surely for sufficiently large n. By concavity of $\log F + \delta$ and the definitions of $y_n^{K^*}(\omega)$ and $x_n^*(\omega)$, it follows that the curve $\log F_n$ lies strictly below the line L^*, joining $(0, \log F_n(x_n^*(\omega))$ and $(K^*, \log F(K^*) + \delta)$. Since $\log F_n(x_n^*(\omega)) \to -\infty$ as $n \to \infty$, eventually the whole of $\log F_n$ within the interval $[M, K - c]$ for some $c > 0$, will lie below the curve $\log F - \delta$, violating (36). This contradiction proves the result. $\square$

In order to prove Theorem 3.2, we need the following lemma, which is adapted from Theorem 2.2 of Tenga and Santer (1984a), and can be proved similarly.

Lemma A.2. *Suppose* $q : [a, b] \mapsto (-\infty, 0]$, $\{b_j\}$, $\{v_j\}$ *and* C_q *are as in Lemma 2.1. Given a convex strictly increasing function* $t : [0, M) \mapsto [0, \infty)$, *where* $M > b$, *define* $q'(x)$ *on* $[a', b'] = [t(a), t(b)]$ *to be the right continuous step function with value* v_j *at* $t(b_j)$ *for* $1 \le j \le n$. *Then the LCM,* $C_{q'}$ *of* q' *satisfies*

$$C_{q'}(t(x)) \le C_q(x) \quad \forall x \in [a, b]. \tag{37}$$

Proof of Theorem 3.2.

Let $F \in \mathcal{L}_p \cap \mathcal{G}$. Define $t : [0,1) \mapsto [0,\infty)$ by

$$t(x) = \begin{cases} (\log F)^{-1}(\log F_p^*(x)) = F^{-1}F_p^*(x) & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly $t(x)$ is strictly increasing. Now, $\log F_p^*(x) = (1-x)\log p$ if $x \in (0,1)$ so that, $t(x) = (\log F)^{-1}((-\log p)(x-1))$ for $x \in (0,1)$. Since F is log-concave, $(\log F)^{-1}$ is convex on $(\log p, 0)$. Consequently $t(x)$ is convex.

If $0 \le x_1 \le \ldots \le x_n < 1$ is an ordered sample from F_p^*, then defining $y_j = t(x_j)$ we have an ordered sample $0 \le y_1 \le \ldots \le y_n < \infty$ from F. (Notice that in this case $b = x_n < 1 = M$) Then by Lemma A.2 it follows that $C_{q'}(t(x_j)) \le C_q(x_j)$ for $1 < j < n$, so that d_n computed from $y_1, \ldots, y_n$ is less than or equal to that computed from $x_1, \ldots, x_n$. This implies that

$$P_{F_p^*}(d_n \ge u) \ge P_F(d_n \ge u) \quad \forall\ u \ge 0$$

We need a couple of lemmas in order to prove Theorem 3.5.

Lemma A.3. *Let $f : [0,\infty) \mapsto [0,1]$ be a nondecreasing function. Let $K \ge 0$ be such that $f(K) = y_0 > 0$. Suppose there exists a sequence of nondecreasing functions $f_n : [0,\infty) \mapsto [0,1]$ such that $\sup_x |f_n(x) - f(x)| \to 0$ as $n \to \infty$. Then,*

$$\sup_{x \ge K} |\log f_n(x) - \log f(x)| \to 0 \qquad \text{as } n \to \infty.$$

Proof. Let $\eta > 0$ be such that $y_0 - \eta > 0$. The function log restricted to the domain $[y_0 - \eta, 1]$ is uniformly continuous.

Suppose $\epsilon > 0$ is given. Then $\exists\, \delta(\epsilon) > 0$ such that $|\log y - \log x| < \epsilon$ if $|y - x| < \delta(\epsilon)$ and $y, x \in [y_0 - \eta, 1]$. Also, since $f_n(K) \to f(K)$, $\exists\, N_\eta$ such that $|f_n(K) - f(K)| < \eta$ for all $n \ge N_\eta$. This implies, in particular, that $f_n(K) > y_0 - \eta$ if $n \ge N_\eta$ (since $y_0 = f(K)$). Since f_n is non-decreasing, if $n \ge N_\eta$ then $f_n(x) > y_0 - \eta\ \forall x \ge K$.

Also, since f is nondecreasing and $f(K) = y_0$, we have, $f(x) > y_0 - \eta \ \ \forall x \ge K$. Since $\sup_x |f_n(x) - f(x)| \to 0$ as $n \to \infty$, $\exists\, N_{\delta(\epsilon)}$ such that if $n \ge N_\eta \vee N_{\delta(\epsilon)}$, then

$\forall x \geq K$, we have, $f_n(x), f(x) \in [y_0 - \eta, 1]$ and $|f_n(x) - f(x)| < \delta(\epsilon)$. Thus, if $n \geq N_\eta \vee N_{\delta(\epsilon)}$, then

$$\sup_{x \geq K} |\log f_n(x) - \log f(x)| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves the result. □

Lemma A.4. *Suppose F_n denotes the empirical distribution function based on a sample of size n from a distribution F belonging to $\mathcal{L}_p$ for any $0 < p < 1$. Then*

$$\sup_{x \geq 0} |\log F_n(x) - \log F(x)| \to 0 \textit{ almost surely } \textit{ as } n \to \infty \tag{38}$$

Proof. Recall that by Glivenko-Cantelli Theorem $\sup_x |F_n(x) - F(x)| \to 0$ almost surely as $n \to 0$. Fix ω such that the convergence takes place. Since F has a point mass p at zero, take $f(x) \equiv F(x)$, $f_n(x) \equiv F_n(x)(\omega)$, $K = 0$ and $y_0 = p$ and apply Lemma A.3 to complete the proof. □

Proof of Theorem 3.5.

Part (a). We prove this in two steps. In the first step we show that under the conditions of the theorem, for every $0 < p < 1$ and $0 < \alpha < 1$, we have, $c_{\alpha,n}(p) \to 0$ as $n \to \infty$.

By Lemma A.2 we have, for all $F \in \mathcal{L}_p$ $(0 < p < 1)$,

$$\sup_{x \geq 0} |\log F_n(x) - \log F(x)| \to 0 \text{ almost surely} \qquad \text{as } n \to \infty.$$

Thus, $\forall \epsilon > 0$,

$$\lim_{m \to \infty} P\left(\sup_{0 \leq x \leq 1} |\log F_n(x) - \log F_p^*(x)| \leq \epsilon/2 \ \forall n \geq m\right) = 1.$$

Given $\epsilon > 0$, and $\alpha > 0$, choose N so that

$$P\left(\sup_{0 \leq x \leq 1} |\log F_n(x) - \log F_p^*(x)| \leq \epsilon/2 \ \forall n \geq N\right) \geq 1 - \alpha.$$

or,

$$P(L(x) \leq \log F_n(x) \leq U(x) \ \forall x \in [0,1] \quad \forall n \geq N) \geq 1 - \alpha \tag{39}$$

where $L(x) = \log F_p^*(x) - \epsilon/2$ and $U(x) = \log F_p^*(x) + \epsilon/2$. Also, $C_{\log F_n}(x) \leq U(x)$ whenever $\log F_n(x) \leq U(x)\ \ \forall x \in [0,1]$ (since $U(x)$ is concave, as $\log F_p^*(x)$ is concave). However, $U(x) - L(x) = \epsilon\ \forall x$. Hence (39) implies

$$
\begin{aligned}
& P\left(\sup_x |\log F_n(x) - C_{\log F_n}(x)| \leq \epsilon \quad \forall n \geq N\right) \geq 1-\alpha \\
\Rightarrow\quad & P\left(\overline{w}\sup_x |\log F_n(x) - C_{\log F_n}(x)| \leq \epsilon\overline{w} \quad \forall n \geq N\right) \geq 1-\alpha \\
\Rightarrow\quad & P(d_n \leq \epsilon\overline{w} \quad \forall n \geq N) \geq 1-\alpha,
\end{aligned}
$$

since $\log F_n(x) \leq L_{\log F_n}(x) \leq C_{\log F_n}(x)$ and since

$$\overline{w}\sup_x |L_{\log F_n}(x) - C_{\log F_n}(x)| \geq d_n.$$

(Here L_q is as defined in the beginning of Section 3). Since $\epsilon > 0$ is arbitrary, by recalling the definition of $c_{\alpha,n}(p)$, we conclude that $c_{\alpha,n}(p) \to 0$ as $n \to \infty$.

In the second step of the proof of part (a), we establish consistency of the test by showing that if $F \in \mathcal{L}_p, (0 < p < 1)$ is not log-concave, then

$$P(d_n \geq c_{\alpha,n}(p)) \to 1 \qquad \text{as } n \to \infty. \tag{40}$$

If $F \in \mathcal{L}_p \cap \mathcal{G}^c$, then there are points $0 \leq x_1 < x_2 < x_3$ and some number $\delta > 0$ such that

$$\log F(x_2) < \frac{x_2 - x_1}{x_3 - x_1}\ \log F(x_3) + \frac{x_3 - x_2}{x_3 - x_1}\ \log F(x_1) - 3\delta. \tag{41}$$

By SLLN and the fact that log is a continuous function we know that for almost all ω, there is $N_1(\omega) < \infty$ such that $\forall n \geq N_1(\omega)$ we have,

$$|\log F_n(x_i) - \log F(x_i)| < \delta \quad \text{for } i = 1, 2, 3. \tag{42}$$

Fix such an ω. Then $\forall n \geq N_1(\omega)$ we have,

$$
\begin{aligned}
C_{\log F_n}(x_2) &\geq \frac{x_2 - x_1}{x_3 - x_1}\ \log F_n(x_3) + \frac{x_3 - x_2}{x_3 - x_1}\ \log F_n(x_1), \\
&\qquad\qquad (\text{ since } C_{\log F_n} \text{ is the LCM of } \log F_n) \\
&> \frac{x_2 - x_1}{x_3 - x_1}\ \log F(x_3) + \frac{x_3 - x_2}{x_3 - x_1}\ \log F(x_1) - \delta \\
&> \log F(x_2) + 2\delta \\
&> \log F_n(x_2) + \delta. \qquad (43)
\end{aligned}
$$

The second and fourth inequalities are due to (42) and the third inequality is by (41).

For every x in the interior of the support of F, there is a sufficiently large n and an integer-valued random variable $i_n(x)$ defined by $X_{i_n(x):n} \leq x < X_{i_n(x)+1:n}$. Further, we have,

$$\log F_n(x) \leq L_{\log F_n}(x) \leq \log F_n(X_{i_n(x)+1:n}) = \log \frac{i_n(x)+1}{n}.$$

Also,

$$0 \leq \log F_n(X_{i_n(x)+1:n}) - \log F_n(x) \leq \log \frac{i_n(x)+1}{n} - \log \frac{i_n(x)}{n}.$$

Since $F(x) > 0 \ \forall x > 0$, by SLLN, it follows that $\forall x > 0$, as $n \to \infty, i_n(x) \to \infty$ almost surely. As a result, we finally get

$$0 \leq L_{\log F_n}(x) - \log F_n(x) \to 0 \ \text{ almost surely as } n \to \infty.$$

Hence, appealing to (43) we can say that, for almost all ω, there is $N(\omega) < \infty$ $(N(\omega) \geq N_1(\omega))$ such that $\forall n \geq N(\omega)$ we have,

$$C_{\log F_n}(x_2) > L_{\log F_n}(x_2) + \delta/2.$$

(Without loss of generality we can take N to be a random variable). This implies that for all $m \geq 1$,

$$\begin{aligned}
P&\left(\sup_x |L_{\log F_m}(x) - C_{\log F_m}(x)| > \delta/2\right) \\
&\geq \quad P(C_{\log F_m}(x_2) - L_{\log F_m}(x_2) > \delta/2) \\
&\geq \quad P(C_{\log F_n}(x_2) - L_{\log F_m}(x_2) > \delta/2 \ \ \forall n \geq m) \\
&\geq \quad P(\{\omega : N(\omega) \leq m\}) \\
&\to \quad 1 \quad \text{as } m \to \infty
\end{aligned}$$

The last convergence result follows since $P(\{\omega : N(\omega) < \infty\}) = 1$ and the event $\{\omega : N(\omega) \leq m\} \uparrow \{\omega : N(\omega) < \infty\}$ as $m \uparrow \infty$.

Now, since

$$d_n \geq \underline{w} \sup_x |L_{\log F_n}(x) - C_{\log F_n}(x)|,$$

the last result implies that $P(d_n \geq \underline{w}\delta/2) \to 1$ as $n \to \infty$. This, together with the fact that $\lim_{n\to\infty} c_{\alpha,n}(p) = 0$ proves (40).

Part (b). We prove this in two steps also. In the first step we show that under the given conditions $c_{\alpha,n}(p) \to 0$ as $n \to \infty$ for all $0 < p < 1$. If we take

$$w^* = \sup_{x \in [0,1]} w(x)$$

then $w^* < \infty$ (since $w : [0,1] \mapsto \mathbb{R}$ is continuous) and for all n,

$$\overline{w} = \sup_j w_j \leq w^*.$$

So now we can proceed as in the first step of the proof of part (a) to make the desired conclusion.

In the second step of the proof we show that if $F \in \mathcal{L}_p$ is not log-concave then (40) holds. As in the previous case, we can find points $0 \leq x_1 < x_2 < x_3$ and a number $\delta > 0$ such that (41) holds. Further, δ can be so chosen that $w(F(x_2)) > 2\delta$. This is because $F(x_2) > 0$ and by assumption $w(x) > 0 \ \forall \ x > 0$.

With the same notations as in the proof of Theorem 3.5, we have for almost all ω, there is $N_1(\omega)$ such that $\forall \ n \geq N_1(\omega)$, (43) holds.

Again, we have

$$\frac{i_n(x_2)+1}{n} = F_n(X_{i_n(x_2)+1:n}) \to F(x_2) \quad \text{almost surely as } n \to \infty.$$

Since w is continuous, it follows that,

$$w_{i_n(x_2)+1} = w\left(\frac{i_n(x_2)+1}{n}\right) \to w(F(x_2)) \quad \text{almost surely as } n \to \infty.$$

Hence, for almost all ω, there is $N_2(\omega)$ such that $\forall \ n \geq N_2(\omega)$, we have

$$w_{i_n(x_2)+1} > w(F(x_2)) - \delta > \delta. \tag{44}$$

Also, since $0 \leq \log F_n(X_{i_n(x_2)+1:n}) - \log F_n(x_2) \leq 1/n$, for all ω, there is $N_3(\omega)$ such that $\forall \ n \geq N_3(\omega)$, we have

$$\log F_n(x_2) > \log F_n(X_{i_n(x_2)+1:n}) - \delta/2. \tag{45}$$

Combining, (43), (44) and (45) we can say that for almost ω if $n \geq N(\omega)$ (where $N \geq \max\{N_1, N_2, N_3\}$ and measurable), then

$$C_{\log F_n}(X_{i_n(x_2)+1:n}) \geq C_{\log F_n}(x_2) > \log F_n(X_{i_n(x_2)+1:n}) + \delta/2$$

and (44) holds. So for almost all ω if $n \geq N(\omega)$, then

$$\begin{aligned} d_n &= \max_{1<j<n} \{w_j(C_{\log F_n}(X_{j:n}) - \log F_n(X_{j:n})\} \\ &\geq w_{i_n(x_2)+1}(C_{\log F_n}(X_{i_n(x_2)+1:n}) - F_n(X_{i_n(x_2)+1:n})) \\ &\geq \delta^2/2. \end{aligned}$$

This implies

$$P(d_n \geq \delta^2/2) \geq P(\omega : N(\omega) \leq n) \to 1 \quad \text{as } n \to \infty.$$

This, together with the fact that $\lim_{n\to\infty} c_{\alpha,n}(p) = 0$ proves (40). □

Proof of Theorem 4.1.

We show that if $0 < p_1 < p_2 < 1$, then for every n,

$$P_{F^*_{p_1}}(d_n \geq u) \geq P_{F^*_{p_2}}(d_n \geq u) \quad \forall\, u \geq 0. \tag{46}$$

This will prove that $c_{\alpha,n}(p)$ increases as $p \downarrow 0$. Since, by definition, $c_{\alpha,n}(p) \leq -\log(1/n) = \log n$ for all p, the result follows.

To prove (46), let us define a (log-concave) distribution $\widetilde{F}_{p_1,p_2,m}$ with point mass p_1 at 0, by

$$\log \widetilde{F}_{p_1,p_2,m}(x) \begin{cases} (1-x)\log p_2 & \text{if } y_m \leq x \leq 1 \\ \log p_1 + \dfrac{\log z_m - \log p_1}{y_m} x & \text{if } 0 \leq x \leq y_m \end{cases}$$

where $0 < y_m < 1$ for $m \geq 1$, $\log z_m = (1 - y_m)\log p_2$ and $y_m \downarrow 0$ as $m \uparrow \infty$.

For each fixed sample size n, let us denote by $U_{1:n} < \cdots < U_{n:n}$, an ordered random sample from $U(0,1)$ distribution. Let $\mathbf{X}^{p_1}$, $\mathbf{X}^{p_2}$ and $\mathbf{X}^m$ (each one is an ordered n-tuple) be defined as follows.

$$X^{p_1}_{i:n} = \begin{cases} 0 & \text{if } U_{i:n} \leq p_1, \\ 1 - \dfrac{\log U_{i:n}}{\log p_1} & \text{if } U_{i:n} > p_1. \end{cases}$$

$$X^{p_2}_{i:n} = \begin{cases} 0 & \text{if } U_{i:n} \leq p_2, \\ 1 - \dfrac{\log U_{i:n}}{\log p_2} & \text{if } U_{i:n} > p_2. \end{cases}$$

$$X^{m}_{i:n} = \begin{cases} 0 & \text{if } U_{i:n} \leq p_1, \\ y_m \left(\dfrac{\log U_{i:n} - \log p_1}{\log z_m - \log p_1}\right) & \text{if } p_1 < U_{i:n} \leq z_m, \\ 1 - \dfrac{\log U_{i:n}}{\log p_1} & \text{if } U_{i:n} > z_m. \end{cases}$$

for $1 \leq i \leq n$. It is easy to check that $\mathbf{X}^{p_1}$, $\mathbf{X}^{p_2}$ and $\mathbf{X}^m$ are ordered samples of size n from $F^*_{p_1}$, $F^*_{p_2}$ and $\widetilde{F}_{p_1,p_2,m}$ respectively.

Notice that $z_m \to p_2$ as $m \to \infty$. Hence for any given $\epsilon > 0$, for sufficiently large m, $z_m < p_2 + \epsilon$.

Let, for sample point ω, the sample realization be $(U_{1:n}(\omega), \ldots, U_{n:n}(\omega))$. Also, let a_n be the function defined in Lemma A.5 (given below). For comparison we consider the following cases :

Case 1. For all $i = 1, \ldots, n$, $U_{i:n}(\omega) > p_2$.

For all sufficiently large m, $U_{i:n}(\omega) > z_m$ for $i = 1, \ldots, n$. This implies, for all sufficiently large m,

$$X^m_{i:n}(\omega) = 1 - \frac{\log U_{i:n}(\omega)}{\log p_2} = X^{p_2}_{i:n}(\omega) \qquad \text{for } i = 1, \ldots, n.$$

Then $a_n(\mathbf{X}^m(\omega)) = a_n(\mathbf{X}^{p_2}(\omega))$.

Case 2. For all $i = 1, \ldots, n$, $U_{i:n}(\omega) \leq p_1$.

Then we have, for all m,

$$X^m_{i:n}(\omega) = 0 = X^{p_2}_{i:n}(\omega) \qquad \text{for } i = 1, \ldots, n.$$

Hence $a_n(\mathbf{X}^m(\omega)) = 0 = a_n(\mathbf{X}^{p_2}(\omega))$.

Case 3. For some $1 \leq k \leq n-1$, $U_{k:n}(\omega) < p_1$ and $U_{k+1:n}(\omega) > p_2$.

It can be checked along the lines of cases 1 and 2 that for sufficiently large m,

$$X^m_{i:n}(\omega) = X^{p_2}_{i:n}(\omega) \qquad \text{for } i = 1, \ldots, n.$$

Hence, again we have $a_n(\mathbf{X}^m(\omega)) = a_n(\mathbf{X}^{p_2}(\omega))$.

Case 4. For some $0 \leq j < k \leq n$, $U_{j:n}(\omega) \leq p_1$ and $p_1 < U_{i:n}(\omega) \leq p_2$ for $j+1 \leq i \leq k$. (We take $U_{0:n} \equiv 0$ and $U_{n+1:n} \equiv 1$).

Thus, for all m, $p_1 < U_{i:n}(\omega) \leq z_m$ for $j+1 \leq i \leq k$. And for all sufficiently large m, $U_{k+1:n}(\omega) > z_m$. Thus for all sufficiently large m we have

$$X^m_{i:n}(\omega) = 0 = X^{p_2}_{i:n}(\omega) \qquad \text{for } i = 1, \ldots, j;$$

$$X_{i:n}^m(\omega) = y_m \left(\frac{\log U_{i:n} - \log p_1}{\log z_m - \log p_1}\right) \qquad \text{for } i = j+1, \ldots, k;$$

and

$$X_{i:n}^m(\omega) = 1 - \frac{\log U_{i:n}(\omega)}{\log p_2} = X_{i:n}^{p_2}(\omega) \qquad \text{for } i = k+1, \ldots, n.$$

Now, check that $a_n(\mathbf{X}^m(\omega)) \geq \widehat{a}_n(\widehat{\mathbf{X}}^m(\omega))$ where

$$\widehat{\mathbf{X}}^m = (X_{k:n}^m, \ldots, X_{n:n}^m)$$

and $\widehat{a}_n(\widehat{\mathbf{X}}^m)$ is the maximum (weighted) difference between the linear interpolant of the points $\{(X_{i:n}^m, \log(i/n)) : i = k, \ldots, n\}$ (call it $\tilde{L}_{n,k}$) and its LCM. Since $X_{i:n}^m(\omega) = X_{i:n}^{p_2}(\omega)$ for $i = k+1, \ldots, n$ and $X_{k:n}^m(\omega) \to 0 = X_{k:n}^{p_2}(\omega)$ as $m \to \infty$, it follows that

$$\widehat{a}_n(\widehat{\mathbf{X}}^m(\omega)) \to a_n(\mathbf{X}^{p_2}(\omega))$$

as $m \to \infty$. This can be checked easily by concentrating our attention to the triangles Δ and Δ_m, where Δ is formed by the points $(0, \log(k/n))$, $(X_{k+1:n}^{p_2}, \log((k+1)/n))$ and $(X_{j^*:n}^{p_2}, \log(j^*/n))$ and Δ_m is formed by the points $(X_{k:n}^m, \log(k/n))$, $(X_{k+1:n}^{p_2}, \log((k+1)/n))$ and $(X_{j^*:n}^{p_2}, \log(j^*/n))$. The index j^* is the smallest index $j \geq k+1$ such that the point $(X_{j:n}, \log(j/n))$ is on the LCM of $\tilde{L}_{n,k}$. From the definition it follows that the triangles Δ_m merge with Δ as $m \to \infty$ and the result follows from the definitions of $a_n(\mathbf{X}^{p_2}(\omega))$ and $\widehat{a}_n(\widehat{\mathbf{X}}^m(\omega))$.

This implies that

$$\limsup_{m\to\infty} a_n(\mathbf{X}^m(\omega)) \geq a_n(\mathbf{X}^{p_2}(\omega)) \tag{47}$$

Now, since $\widetilde{F}_{p_1,p_2,m}$ is a log-concave distribution with point mass p_1 at zero, an application of Lemma A.2 (as in the proof of Theorem 3.2) shows that $\forall\, m$, for almost all ω, we have,

$$a_n(\mathbf{X}^{p_1}(\omega)) \geq a_n(\mathbf{X}^m(\omega)).$$

This, together with (47) and the other three cases studied above imply that

$$a_n(\mathbf{X}^{p_1}(\omega)) \geq a_n(\mathbf{X}^{p_2}(\omega)) \qquad \text{almost surely.}$$

So we have proved (46). $\square$

In order to prove Theorem 4.2, we need the following lemma, which is not very difficult to prove.

Lemma A.5. *Let*

$$S = \{(x_1, \ldots, x_n) \,:\, x_i \in \mathbb{R}\ \forall\, i, x_1 \leq x_2 \leq \cdots \leq x_n\},$$

and $a_n \,:\, S \mapsto \mathbb{R}$ *be the function that maps* $(X_{1:n}, \ldots, X_{n:n})$ *into* d_n *(that is,* $d_n = a_n(X_{1:n}, \ldots, X_{n:n})$*).*

(a) The points of discontinuity of the map a_n *are contained in the set*

$$D(a_n) = \{(x_1, \ldots, x_n) : x_1 \leq \cdots \leq x_i = x_{i+1} \leq \cdots \leq x_n \text{ for some } 1 \leq i \leq n-1\}$$

(b) If F *is log-concave and does not have a point mass at 0, then the set* $D(a_n)$ *has probability 0 under* F*.*

Proof of Theorem 4.2.

By Theorem 3.2 and Theorem 4.1, the result holds if F has a point mass at 0. So suppose F is continuous throughout and does not have a point mass at 0. Then there is $x_0 > 0$ such that $\log F$ is strictly concave on $(0, x_0)$ (since $\log F(x) \downarrow -\infty$ as $x \downarrow 0$ and $\log F$ is concave) and hence it is almost everywhere differentiable. Further, there is a decreasing sequence of points $\{r_m \,:\, m = 1, 2, \ldots\}$ such that $r_1 < x_0$ and $\lim_{m\to\infty} r_m \to 0$, such that the tangents to $\log F$ passing through the points $(r_m, \log F(r_m))$ have slopes increasing to infinity (see Figure 3). Suppose that the tangent passing through $(r_m, \log F(r_m))$ intersects the line $x = 0$ at the point $(0, z_m)$. It follows that $z_m \downarrow -\infty$ as $m \to \infty$. Define $p_m = e^{z_m}$.

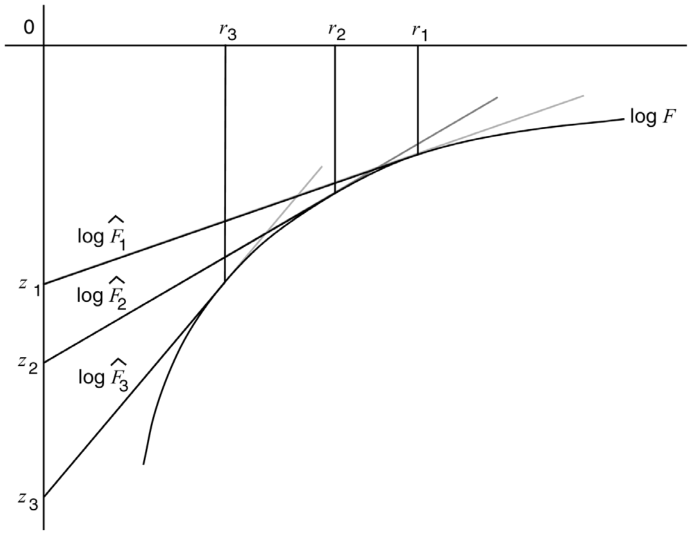

Figure 3. Construction used in the Proof of Theorem 4.2.

Define distributions $\widehat{F}_m$ by

$$\log \widehat{F}_m(x) = \begin{cases} \log F(x), & \text{if } x \geq r_m, \\ \text{tangent line joining } (0, \log p_m) \text{ and} & \\ \qquad (r_m, \log F(r_m)) \text{ evaluated at } x, & \text{if } 0 \leq x < r_m, \\ -\infty, & \text{if } x < 0. \end{cases}$$

So, $\widehat{F}_m$ is a log-concave distribution with a point mass p_m at 0.

Thus, by Theorem 3.2,

$$P_{F^*_{p_m}}(d_n \geq u) \geq P_{\widehat{F}_m}(d_n > u) \qquad \forall\, u \geq 0,$$

Notice that $\widehat{F}_m \implies F$ as $m \to \infty$. Since F is continuous everywhere, by Lemma A.5(b) we get

$$P_{\widehat{F}_m}(d_n > u) \to P_F(d_n > u) \qquad \forall\, u \geq 0$$

as $m \to \infty$.

On the other hand, by Theorem 4.1 we know that for every $m \geq 1$, $P_{F^*_{p_m}}(d_n \geq c_{\alpha,n}) \leq \alpha$. Also this probability increases as $m \to \infty$ (refer to (46)). As a result we have

$$\alpha \geq P_F(d_n > c_{\alpha,n}). \qquad \square$$

Proof of Lemma 5.2.

Parts (a)–(c) follow from the definition of the TTF transform. If either of the conditions of parts (b) and (c) hold, then F is not log-concave and T_F^{-1} is not convex. Now suppose that F is continuous in $(a, b]$ and T_F^{-1} is continuous in $[0, 1)$. It can be verified that for all $x > a$, the left-derivative of $\log F$ at x is equal to the reciprocal of the left-derivative of T_F^{-1} at $F(x)$. Further, the right-derivative of $\log F$ at x is equal to the reciprocal of the right-derivative of T_F^{-1} at $F(x)$, as long as $T_F^{-1}(F(x)) < 1$. The result of part (d) in the special case of F having no point mass in $(a, b]$ is proved by observing that a continuous function is convex (concave) if and only if its left- and right-derivatives are non-decreasing (non-increasing). The proof is completed by observing that if F has a jump discontinuity at $x_0 \in (a, b]$, then $\log F$ is not concave at x_0 and T_F^{-1} is not convex at $F(x_0-)$. $\square$

Proof of Theorem 5.4.

Part (a). For $t \in [0, 1]$, and $F \in \mathcal{L}_B$ (without loss of generality let us take $a = 0$, that is, the support of the distribution F is $[0, b]$ for some $b > 0$),

$$
\begin{aligned}
& \left| \int_0^{F^{-1}(t)} F(u)du - \int_0^{F_n^{-1}(t)} F_n(u)du \right| \\
\leq \; & \int_0^{F^{-1}(t)} |F(u) - F_n(u)|du + \left| \int_{F_n^{-1}(t)}^{F^{-1}(t)} F_n(u)du \right| \\
\leq \; & \int_0^{F^{-1}(t)} |F(u) - F_n(u)|du + |F_n^{-1}(t) - F^{-1}(t)| \\
\leq \; & F^{-1}(1) \sup_x |F_n(x) - F(x)| + \sup_{t\in[0,1]} |F_n^{-1}(t) - F^{-1}(t)|
\end{aligned}
$$

By Glivenko-Cantelli Theorem the first term in the inequality converges to 0 almost surely. In order to complete the proof, we have to show that the second term also converges to 0 almost surely, under the assumption that F^{-1} is continuous on $[0, 1]$ and F is strictly increasing on $[0, b]$.

Since, F^{-1} is continuous on $[0, 1]$, given $\epsilon > 0$, $\exists\, \delta_\epsilon > 0$ such that if $x, y \in [0, 1]$, $|x - y| \leq \delta_\epsilon$, then $|F^{-1}(x) - F^{-1}(y)| \leq \epsilon$.

We have,

$$P(|F_n^{-1}(t) - F^{-1}(t)| > \epsilon)$$

$$
\begin{aligned}
&= P(|X_{[nt]:n} - F^{-1}(t)| > \epsilon) \text{ where } [nt] \text{ is smallest integer } \geq nt \\
&= P(|F^{-1}(U_{[nt]:n}) - F^{-1}(t)| > \epsilon) \ \text{ since } F^{-1} \text{ is continuous on } [0,1] \\
&\leq P(|U_{[nt]:n} - t| > \delta_\epsilon),
\end{aligned}
$$

where $U_{i:n}$ denotes the i-th smallest order statistics for a random sample of size n from Uniform(0,1) distribution. Since $U_{k:n}$ has Beta$(k, n-k+1)$ distribution, and mean of Beta$(k, n-k+1)$ is k/n and $|[nt] - nt| \leq 1$, hence by taking fourth moment and using Chebyshev's inequality we can bound the last term in the above inequality by $C(t)\delta_\epsilon^{-4}n^{-2}$ where $C(t)$ is a constant depending on t and uniformly bounded on $[0,1]$. Now applying Borel-Cantelli lemma we obtain

$$
P(|F_n^{-1}(t) - F^{-1}(t)| > \epsilon \ \text{ infinitely often}) = 0,
$$

which proves that $F_n^{-1}(t)$ converges to $F^{-1}(t)$ pointwise, almost surely. The pointwise a.s. convergence of a sequence of monotone random functions to a bounded and continuous function on a compact interval is equivalent to their uniform a.s. convergence. Hence, $\sup_{t\in[0,1]} |F_n^{-1}(t) - F^{-1}(t)| \to 0$ a.s. as $n \to \infty$.

Thus we have actually proved that

$$
\lim_{n\to\infty} \sup_{t\in[0,1]} \left| \int_0^{F^{-1}(t)} F(u)du - \int_0^{F_n^{-1}(t)} F_n(u)du \right| = 0 \quad \text{almost surely.}
$$

Since $F_n^{-1}(1) \to b$ almost surely as $n \to \infty$, the result follows.

Part (b). Since $\widetilde{T}_n^{-1}$ is the linear interpolation between the successive points $(k/n, S_{k:n})$, it is nondecreasing and hence for $\frac{k-1}{n} \leq t \leq \frac{k}{n}$, we have

$$
T_n^{-1}\left(\frac{k-1}{n}\right) \leq \widetilde{T}_n^{-1}(t) \leq T_n^{-1}\left(\frac{k}{n}\right).
$$

Consequently, the result follows from part (a). $\square$

Proof of Theorem 5.8. We only prove (a) for the case when F has at most one point mass, and the possible point mass is at 0. (b) easily follows from (a) when F is also log-concave.

If F has a jump discontinuity at x_0 then T_F has a jump discontinuity at

$$z(x_0) := T_F^{-1}(F(x_0)) = \frac{\int_0^{x_0} F(u)du}{\int_0^{F^{-1}(1)} F(u)du},$$

and the size of the jump is the same as that of F.

First fix $t \in (0,1]$. Without loss of generality we suppose $X_{n-1:n} > 0$ (true almost surely as $n \to \infty$). Then $\exists 1 \leq k_n(t) \leq n-1$ such that $\frac{k_n(t)}{n} \leq \tilde{T}_n(t) \leq \frac{k_n(t)+1}{n}$, whereby,

$$T_n^{-1}\left(\frac{k_n(t)}{n}\right) = S_{k_n(t);n} \leq t \leq S_{k_n(t)+1;n} = T_n^{-1}\left(\frac{k_n(t)+1}{n}\right). \tag{48}$$

We now show that $\frac{k_n(t)}{n} \to T_F(t)$ a.s. By Theorem 5.4(a) and (48) we have

$$T_F^{-1}\left(\frac{k_n(t)}{n}\right) \to t, \quad \text{a.s.}$$

From this the result follows since, by the assumption that F^{-1} is continuous on $[0,1]$, we have $\frac{k_n(t)}{n} = T_F\left(T_F^{-1}\left(\frac{k_n(t)}{n}\right)\right)$ and T_F is continuous on $(0,1]$ by the observation made above. Thus, $\tilde{T}_n(t) \to T_F(t)$ a.s.

Now suppose $t = 0$. Then, if F does not have a point mass at 0, then w.p. 1, $\tilde{T}_n(0) = \frac{1}{n} \to 0 = T_F(0)$. So, suppose that F has a point mass at zero and $F(0) = p$. Then $T_F(0) = p$. Notice that $\tilde{T}_n(0) = F_n(0) + \frac{1}{n}$. Since $F_n(0) \to F(0) = p$, a.s., we have $\tilde{T}_n(0) \to T_F(0)$ a.s.

Thus for all $t \in [0,1]$, $\tilde{T}_n(t) \to T_F(t)$ a.s. Since $\tilde{T}_n$ is a distribution function on $[0,1]$ for every sample realization, and since the pointwise limit T_F is also a distribution function on $[0,1]$, and under the assumption both have only one point mass at 0, (by mimicking the proof of Glivenko-Cantelli theorem) we conclude that

$$\sup_{t\in[0,1]} |\tilde{T}_n(t) - T_F(t)| \to 0 \quad \text{a.s., as} \quad n \to \infty. \qquad \square$$

In order to prove Theorem 6.1 we need the following lemma.

Lemma A.6. *Let $s_{1;n}, \ldots, s_{n;n}$ be the values of $S_{1;n}, \ldots, S_{n;n}$ defined by (17) for a given set of order statistics $0 \leq x_{1:n} \leq \ldots \leq x_{n;n} < \infty$. Suppose $t : [0, M) \mapsto [0, \infty)$*

is a concave increasing function for some $M > b$. Also, let $s'_{1;n}, \ldots, s'_{n;n}$ be the values of $S_{1;n}, \ldots, S_{n;n}$ defined by (17) when $x_{k:n}$ is replaced by $t(x_{k:n})$ for $k = 1, \ldots, n$. Then we can construct a concave increasing function $\xi : [0,1] \mapsto [0,1]$ such that $\xi(s_{k;n}) = s'_{k;n}$ for $k = 1, \ldots, n$.

Proof: Define $\xi : [0,1] \to [0,1]$ by $\xi(s_{k;n}) = s'_{k;n}$ and by linear interpolation between the points $\{s_{k;n}, k = 0, 1, \ldots, n\}$. Since t is increasing, it follows that $s'_{k;n}$ are nondecreasing in k and hence, ξ is non-decreasing. W.l.o.g. we may assume that $x_{k:n}$ are all different. Then, in order to prove the concavity of ξ, it is enough to prove that for every $1 \leq k \leq n-2$, we have

$$\frac{\xi(s_{k+1;n}) - \xi(s_{k;n})}{s_{k+1;n} - s_{k;n}} \geq \frac{\xi(s_{k+2;n}) - \xi(s_{k+1;n})}{s_{k+2;n} - s_{k+1;n}} \tag{49}$$

By definition of ξ and $s_{k;n}$, (49) can be re-written as

$$\frac{t(x_{k+1:n}) - t(x_{k:n})}{x_{k+1:n} - x_{k:n}} \geq \frac{t(x_{k+2:n}) - t(x_{k+1:n})}{x_{k+2:n} - x_{k+1:n}},$$

which is obviously true since t is a concave increasing function. □

Proof of Theorem 6.1. The stated result can be proved along the lines of the proof of Theorem 3.2, by taking $t(x) = F_p^{*-1}(F(x))$ in Lemma A.6. □

Proof of Theorem 6.2.

The proof follows along the lines of that of Theorem 4.1, once we recognize the following fact which is analogous to Lemma A.5: the discontinuity points of the test statistic t_n are contained in the set

$$\begin{aligned} D(t_n) &= \{(X_{1:n}, \ldots, X_{n:n}) : X_{1:n} = \cdots = X_{i:n} < X_{i+1:n} \leq \cdots \leq \\ &\qquad X_{j-1:n} = X_{j:n} \leq \cdots \leq X_{n:n} \quad \text{for some } \ 1 \leq i < j-1 < n\}. \end{aligned}$$

We omit the details. □

Proof of Theorem 6.3.

The proof follows along the lines of that of Theorem 4.2. □

Proof of Lemma 6.4.

Let $X_1, \dots, X_n$ be samples from F_p^*. Then for $i = 1, \dots, n$, $X_i = F_p^{*-1}(U_i)$ where $U_1, \dots, U_n$ are i.i.d. $U(0,1)$. Thus,

$$X_i = \begin{cases} 0 & \text{if } 0 \le U_i < p, \\ 1 - \dfrac{\log U_i}{\log p} & \text{if } p \le U_i \le 1 \end{cases}.$$

Consequently, $X_{i:n} = 1 - \frac{\log U_{i:n}}{\log p}$ for all $i = 1, \dots, n$ if and only if $U_{1:n} \ge p$. Hence, from (17) it follows that

$$\mathbf{S_p} = (S_{1:n}, \dots, S_{n:n}) = (\widetilde{Z}_{1:n}, \dots, \widetilde{Z}_{n,n}) = \widetilde{\mathbf{Z}}$$

if and only if $U_{1:n} \ge p$, where,

$$\widetilde{Z}_{k;n} = \frac{\sum_{j=1}^{k} (\log U_{k:n} - \log U_{j:n})}{\sum_{j=1}^{n} (\log U_{n:n} - \log U_{j:n})} \quad k = 1, \dots, n. \tag{50}$$

Let us define for fixed n, $B_p = \{\omega : U_{1:n}(\omega) \ge p\}$. Then, $B_{p'} \subseteq B_p$ for $0 < p < p' < 1$. Also, $P(B_p) = (1-p)^n \uparrow 1$ as $p \downarrow 0$. Hence, for any $A \in \mathcal{B}(\mathcal{R}^n)$ (that is, the Borel σ-algebra on $\mathcal{R}^n$) we have,

$$\begin{aligned} P(\mathbf{S_p} \in A) &= P(\mathbf{S_p} \in A \cap B_p) + P(\mathbf{S_p} \in A \cap B_p^c) \\ &= P(\widetilde{\mathbf{Z}} \in A \cap B_p) + P(\mathbf{S_p} \in A \cap B_p^c), \end{aligned}$$

where the first term on right hand side converges to $P(\widetilde{\mathbf{Z}} \in A)$ and the second term converges to 0 as $p \downarrow 0$.

If U_i follows $U(0,1)$ distribution, then $E_i = -\log(1 - U_i)$ has unit exponential distribution. Hence it follows from (31) that $\widetilde{\mathbf{Z}}$ has the same joint distribution as that of $\mathbf{Z}$ and this completes the proof. □

Proof of Theorem 6.5.

It can be easily checked, as in the case of t_n, that for every n, the set of discontinuity points of $\widetilde{t}_n$, has measure 0. The stated result follows from Lemma 6.4. □

Proof of Theorem 6.6.

The key step in showing the consistency of the test is to show that

$$\sup_{x\in[0,1]} |V_n(x) - x| \to 0 \quad \text{almost surely} \qquad \text{as } n \to \infty. \tag{51}$$

By virtue of (51) we immediately deduce that $\tau_{\alpha,n} \to 0$ as $n \to \infty$. Now the rest of the proof follows from the characterization of log-concave distributions through their TTF transform, Theorem 5.8, and using the same technique as in the proof of Theorem 3.5.

We observe that since $V_n(x)$ is a nondecreasing function for each sample point, and [0,1] is a compact set, in order to prove (51) it is enough to prove point-wise almost sure convergence. Since $V_n(0) \equiv 0$ and $V_n(1) \equiv 1$, we only show that for all $x \in (0,1)$, $V_n(x) \to x$ almost surely. Fix $x \in (0,1)$. There exists a sequence of integers $i_n(x)$ satisfying the property that $\frac{i_n(x)}{n} \le x < \frac{i_n(x)+1}{n}$. Clearly, $\frac{i_n(x)}{n} \to x$ as $n \to \infty$. By Lemma A.7 given below, we have

$$\begin{aligned} V_n\left(\frac{i_n(x)}{n}\right) &= Z_{i_n(x):n} &&\to x \text{ almost surely,} \\ \text{and } V_n\left(\frac{i_n(x)+1}{n}\right) &= Z_{i_n(x)+1:n} &&\to x \text{ almost surely.} \end{aligned}$$

Since V_n interpolates linearly between points $(Z_{k:n}, k/n)$, the result follows. □

Lemma A.7. *Let $x \in (0,1)$ and $i_n(x)$ be a sequence of integers such that $\frac{i_n(x)}{n} \to x$ Then*

$$Z_{i_n(x):n} \to x \quad \textit{almost surely.}$$

Proof: From the definition of $Z_{k:n}$ we have,

$$Z_{i_n(x):n} = \frac{i_n(x)}{n} \frac{\frac{1}{i_n(x)} \sum_{j=n-i_n(x)+1}^{n} (E_{j:n} - E_{n-i_n(x)+1:n})}{\frac{1}{n}\sum_{j=1}^{n} E_{j:n} - E_{1:n}}$$

We observe that,

$$\frac{1}{n}\sum_{j=1}^{n} E_{j:n} = \frac{1}{n}\sum_{j=1}^{n} E_j \to 1 \quad \text{almost surely,}$$

$$E_{1:n} \to 0 \quad \text{almost surely.}$$

Hence, it is enough to show that if $\frac{i_n(x)}{n} \to x$, then $\frac{1}{i_n(x)} \sum_{j=n-i_n(x)+1}^{n} (E_{j:n} - E_{n-i_n(x)+1:n})$ $\to 1$ almost surely.

Recall the following result about exponential distributions. Let $X_1, \ldots, X_n$ be i.i.d. unit exponential random variables. Then for $1 \leq j \leq n-2$, the random vector $(X_{j+2:n} - X_{j+1:n}, \ldots, X_{n:n} - X_{j+1:n})$ is independent of $X_{j+1:n}$ and has the same joint distribution as that of the order statistics $(Y_{1:n-j+1}, \ldots, Y_{n-j+1:n-j+1})$, where $\{Y_i, i = 1, \ldots, n-j+1\}$ are i.i.d. unit exponential random variables.

Applying this result to our setting we have,

$$\sum_{j=n-i_n(x)+2}^{n} (E_{j:n} - E_{n-i_n(x)+1:n}) \stackrel{\mathcal{D}}{=} \sum_{j=1}^{i_n(x)-1} Y_{j:i_n(x)-1}.$$

We now show that given any $\epsilon > 0$,

$$P\left(\left|\frac{1}{i_n(x)-1} \sum_{j=n-i_n(x)+2}^{n} (E_{j:n} - E_{n-i_n(x)+1:n}) - 1\right| > \epsilon \ \text{ i.o.}\right) = 0.$$

Consider,

$$\begin{aligned}
& P\left(\left|\frac{1}{i_n(x)-1} \sum_{j=n-i_n(x)+2}^{n} (E_{j:n} - E_{n-i_n(x)+1:n}) - 1\right| > \epsilon\right) \\
= \ & P\left(\left|\frac{1}{i_n(x)-1} \sum_{j=1}^{i_n(x)-1} Y_{j:i_n(x)-1} - 1\right| > \epsilon\right) \\
= \ & P(|W_n(x) - (i_n(x)-1)| > \epsilon(i_n(x)-1)), \\
& \qquad \text{where } \ W_n(x) \sim \Gamma(1, i_n(x)-1) \\
= \ & P(|W_n(x) - E(W_n(x))| > \epsilon k_n(x)) \ \text{ where we write } k_n(x) = i_n(x) - 1 \\
\leq \ & \frac{1}{\epsilon^4 k_n^4} E|W_n(x) - E(W_n(x))|^4 \\
= \ & \frac{3k_n(x)(k_n(x)+2)}{\epsilon^4 k_n^4(x)} \\
\leq \ & \frac{4}{\epsilon^4} \cdot \frac{1}{k_n^2(x)}.
\end{aligned}$$

For any $0 < \delta < x$ we have, for all sufficiently large n, $k_n > n(x-\delta)$, which implies that $\frac{1}{k_n^2(x)} < \frac{1}{n^2(x-\delta)^2}$. Hence,

$$\sum_{n=1}^{\infty} P\left(\left|\frac{1}{i_n(x)-1}\sum_{j=n-i_n(x)+2}^{n}(E_{j:n} - E_{n-i_n(x)+1:n}) - 1\right| > \epsilon\right) < \infty.$$

So an application of Borel-Cantelli lemma gives the result. □

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