(8-4) a) $\begin{array}{|c|ccc|c|} \hline X & 1 & 2 & 3 & p(X) \\ \hline 1 & .1 & .2 & 0 & .3 \\ 2 & .1 & 0 & .2 & .3 \\ 3 & 0 & .1 & .3 & .4 \\ \hline p(Y) & .2 & .3 & .5 & \\ \hline \end{array}$

b) \[ p(X \mid Y = 2) = \frac{p(X, 2)}{p_Y(2)} \]
\[ \begin{array}{c} \\ \hline 1 \quad 2 \\ \hline .2 & 2 \\ .3 & 3 \\ \hline \end{array} \]
\[ \begin{array}{c} \\ \hline 2 \quad 0 \\ \hline .3 & 0 \\ \hline \end{array} \]
\[ \begin{array}{c} \\ \hline 3 \quad 1 \\ \hline .3 & 1 \\ \hline \end{array} \]

c) \[ p(Y \mid X = 3) = \frac{p(3, Y)}{p_X(3)} \]
\[ \begin{array}{c} \\ \hline 1 \quad 0 \\ \hline .4 & 0 \\ \hline \end{array} \]
\[ \begin{array}{c} \\ \hline 2 \quad 1 \\ \hline .4 & 1 \\ \hline \end{array} \]
\[ \begin{array}{c} \\ \hline 3 \quad 3 \\ \hline .4 & 3 \\ \hline \end{array} \]

(8-4) d) Is $p(X, Y) = p(X) \cdot p(Y)$ for each $(X, Y)$? Consider $(1, 1)$:

\[ \text{Is } p(1, 1) = p_X(1) \cdot p_Y(1)? \]
\[ .1 \neq .3 \times .2 \quad \text{NO!} \]

Thus, $X$ and $Y$ are not independent.
First, find $p(X \mid Y = 0)$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$p(X \mid Y = 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
</tr>
</tbody>
</table>

So $E(X \mid Y = 0) = \sum_X X p(X \mid Y = 0) = 1(0.4) + 2(0.2) + 3(0.4) = 2.0$

b) $X$ | $p(X \mid Y = 1)$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
</tr>
</tbody>
</table>

So $E(X \mid Y = 1) = \sum_X X p(X \mid Y = 1) = 1(0.6) + 2(0.2) + 3(0.2) = 1.6$

c) For $Y = 1$, we expect a smaller value of $X$ (1.6) than when $Y = 0$ (2.0). This suggests a negative relationship between $X$ and $Y$.

d) Is $p(X, Y) = p(X) p(Y)$ for each $(X, Y)$? Consider $(1, 0)$:

Is $p(1, 0) = p_X(1) p_Y(0)$?

$.2 \neq .5 \times .5$  NO!

Since it is not the case that $p(X, Y) = p(X) p(Y)$ for each $(X, Y)$, we know that $X$ and $Y$ are not independent.
(8-15) a) \((X,Y)\) \(p(X,Y)\)  \(W = [X - E(X) \{Y - E(Y)\}]\)

\[\begin{array}{ccc}
(1,1) & 0.2 & (1 - 1.5)(1 - 1.5) = 0.25 \\
(1,2) & 0.3 & (1 - 1.5)(2 - 1.5) = -0.25 \\
(2,1) & 0.3 & (2 - 1.5)(1 - 1.5) = -0.25 \\
(2,2) & 0.2 & (2 - 1.5)(2 - 1.5) = -0.25 \\
\end{array}\]

Note \(E(X) = E(Y) = 1.5\)

Thus, \(p(W)\) is

\[
\begin{array}{cc}
W & p(W) \\
-0.25 & 0.6 \\
0.25 & 0.4 \\
\end{array}
\]

b) \(E(W) = \sum \sum \ W \ p(W) = (-0.25)(0.6) + (0.25)(0.4) = -0.05\)

c) \(COV(X,Y) = \sum \sum \ XY \ p(X,Y) - E(X) E(Y)\)

\[= (1)(1)(0.2) + (1)(2)(0.3) + (2)(1)(0.3) + (2)(2)(0.2) - (1.5)^2\]

\[= 0.2 + 0.6 + 0.6 + 0.8 - 2.25 = -0.05\]

d) They are the same, as they should be since

\(COV(X,Y) = E[ (X - E(X))(Y - E(Y)) ]\).
\( \text{(8-19) a) } \)

\[
\begin{array}{c|ccc|c}
X & 0 & 1 & 2 & p(X) \\
\hline
0 & .1 & 0 & .1 & .2 \\
1 & 0 & .3 & 0 & .3 \\
2 & 0 & .3 & 0 & .3 \\
3 & .1 & 0 & .1 & .2 \\
\end{array}
\]

\[
p(Y) = .2 \quad .6 \quad .2
\]

\[
E(X) = 1.5 \quad \quad E(Y) = 1
\]

\[
\text{COV} (X, Y) = 1(1)(.3) + 2(1)(.3) + 3(2)(.1) - (1.5)(1)
\]
\[
= .3 + .6 + .6 - 1.5 = 0
\]

b) \( X \) and \( Y \) are independent if and only if \( p(X,Y) = p(X) \cdot p(Y) \) for each \((X,Y)\).

Consider \((0,0)\):

\[
\text{Is } p(0,0) = p_X(0) \cdot p_Y(0) ?
\]
\[
.1 \neq .2 \times .2 \quad \quad \text{NO!}
\]

Since it is not the case that \( p(X,Y) = p(X) \cdot p(Y) \) for each \((X,Y)\), \( X \) and \( Y \) are not independent.

c) \( E(X \mid Y = 0) = 1.5; \ E(X \mid Y = 1) = 1.5; \ E(X \mid Y = 2) = 1.5 \). 
\( E(Y \mid X = 0) = 1; \ E(Y \mid X = 1) = 1; \ E(Y \mid X = 2) = 1; \ E(Y \mid X = 3) = 1 \).

Thus, \( E(X \mid Y) \) is invariant to the value of \( Y \), and \( E(Y \mid X) \) is invariant to the value of \( X \). What we expect \( X(Y) \) to be as \( Y(X) \) changes does not change. This suggests that \( \text{COV}(X,Y) = 0 \) as seen above in part (a). But there is some sort of relationship between \( X \) and \( Y \), as seen in part (b); this is just not the type of relationship which the covariance uncovers.
Note that for each possible \((X,Y)\), we have \(Y = X + 1\).

\[\text{b) } \quad E(X) = 2.1 \quad E(Y) = 3.1\]

\[\begin{align*}
\sigma_X^2 &= \left[ \sum X^2 p(X) \right] - [E(X)]^2 = (1)^2(0.3) + (2)^2(0.3) + (3)^2(0.1) - (2.1)^2 = 0.69 \\
\sigma_Y^2 &= (2)^2(0.3) + (3)^2(0.3) + (4)^2(0.1) - (3.1)^2 = 0.69 \\
\text{cov}(X,Y) &= \left[ \sum \sum X \cdot Y \cdot p(X,Y) \right] - E(X)E(Y) \\
&= 1(2)(0.3) + 2(3)(0.3) + 3(4)(0.1) - (2.1)(3.1) = 0.69 \\
\rho &= \frac{\text{COV}(X,Y)}{\sigma_X \sigma_Y} = \frac{0.69}{0.69 \cdot 0.69} = 1
\end{align*}\]

c) Since \(Y = X + 1\) for each possible \((X,Y)\), there is a perfect linear (and positive) relationship between \(X\) and \(Y\). Thus, \(\rho = 1\).
\[ (8.25) \quad \begin{array}{ccccccc}
X & Y & XY & (X_i - \bar{X})^2 & (Y_i - \bar{Y})^2 \\
\hline
2 & 3 & 6 & 4 & 1 \\
3 & 3 & 9 & 1 & 1 \\
5 & 4 & 20 & 1 & 0 \\
4 & 3 & 12 & 0 & 1 \\
4 & 4 & 16 & 0 & 0 \\
6 & 7 & 42 & 4 & 9 \\
\hline
\bar{X} = 4 & \bar{Y} = 4 & \Sigma = 105 & \Sigma = 10 & \Sigma = 12 \\
\end{array} \]

\[
\text{Cov}(X,Y) = \frac{1}{n-1} \left[ (\Sigma X_iY_i - n(\bar{X})(\bar{Y})) \right] = (1/5) \left[ 105 - (6)(4)(4) \right] = 1.8
\]

\[
s_X^2 = \frac{1}{n-1} \Sigma (X_i - \bar{X})^2 = (1/5)(10) = 2
\]

\[
s_Y^2 = \frac{1}{n-1} \Sigma (Y_i - \bar{Y})^2 = (1/5)(12) = 2.4
\]

\[
\hat{\rho} = \frac{\text{Cov}(X,Y)}{s_X s_Y} = \frac{1.8}{\sqrt{2} \sqrt{2.4}} = .822
\]

\[ (8.31) \quad E(W) = 20000 \quad E(A) = 3000 \quad \rho = .4 \]

\[ \sigma_W = 5000 \quad \sigma_A = 2500 \]

\[ \text{Tl} = W + A \]

\[ E(Tl) = E(W) + E(A) = 20000 + 3000 = $23,000 \]

(continued)
\[ \sigma_{W}^{2} = \sigma_{A}^{2} + \sigma_{W}^{2} + 2 \text{COV} (W,A) \]

But \[ \rho = \frac{\text{COV} (W,A)}{\sigma_{W} \sigma_{A}} \]

\[ 0.4 = \frac{\text{COV} (W,A)}{(5000)(2500)} \]

\[ \text{COV} (W,A) = 5,000,000 \]

So,

\[ \sigma_{W}^{2} = (5000)^{2} + (2500)^{2} + 2(5,000,000) = 41,250,000 \]

\[ \sigma_{W} = 6422.62 \]

(8-35) \[ \text{COV} (X,Y) = \text{E}(XY) - \text{E}(X) \text{E}(Y) \]

\[ 1 = \text{E}(XY) - (2)(3) \]

\[ \text{E}(XY) = 7 \]

(8-37) The hint tells us that \( W \) is normal. Note

\[ \text{E}(W) = \text{E}(X) + \text{E}(Y) = 5 + 3 = 8 \] (continued)

\[ \sigma_{W}^{2} = \sigma_{X}^{2} + \sigma_{Y}^{2} = 25 + 16 = 41 \]

Thus, \( W \sim N(8,41) \)

So,

\[ \text{Pr} (W > 9) = \text{Pr} \left( \frac{W - \text{E}(W)}{\sigma_{W}} > \frac{9 - 8}{\sqrt{41}} \right) = \text{Pr} (Z > .16) = .4364 \]

(8-38) \((X + Y) \geq 4\) if \((X,Y)\) equals \((1,3), (2,2),\) or \((2,3).\) Since these events are mutually exclusive, we sum their probabilities to determine \[ \text{Pr} (X + Y \geq 4) = .2. \]